

$$2. T: P_2 \rightarrow P_1 \quad S: P_1 \rightarrow P_2 \quad S(p(x)) = \int_0^x p(t) dt$$

$$* = \{1, x\} \quad y = \{1, x, x^2\}$$

$$a) [T]_{*y}$$

$$\begin{array}{l} \left(\begin{array}{l} 1 = a_{11}(1) + a_{21}(x) \\ 0 = a_{11}(1) + a_{21}(x) \end{array} \right. \quad \begin{array}{l} x = a_{12}(1) + a_{22}(x) \\ 1 = a_{12}(1) + a_{22}(x) \end{array} \quad \begin{array}{l} x^2 = a_{13}(1) + a_{23}(x) \\ 2x = a_{13}(1) + a_{23}(x) \end{array} \\ \text{differentiate} \quad \begin{array}{l} 0 \\ 0 \end{array} \quad \begin{array}{l} 0 \\ 1 \end{array} \quad \begin{array}{l} 0 \\ 0 \end{array} \quad \begin{array}{l} 1 \\ 0 \end{array} \quad \begin{array}{l} 0 \\ 2 \end{array} \end{array}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$[S]_{y*}$$

$$\left(\begin{array}{l} 1 = b_{11}(1) + b_{21}(x) + b_{31}(x^2) \\ x = b_{11}(1) + b_{21}(x) + b_{31}(x^2) \end{array} \right. \quad \begin{array}{l} x = b_{12}(1) + b_{22}(x) + b_{32}(x^2) \\ \frac{1}{2}x^2 = b_{12}(1) + b_{22}(x) + b_{32}(x^2) \end{array} \\ \text{integrate} \quad \begin{array}{l} 0 \\ 1 \\ 0 \end{array} \quad \begin{array}{l} 1 \\ 0 \\ 0 \end{array} \quad \begin{array}{l} 0 \\ 0 \\ \frac{1}{2} \end{array}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$b) [T]_{*y} [S]_{y*} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[S]_{y*} [T]_{*y} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{From Problem 1, } [T]_{*y} [S]_{y*} = [T \circ S]_{**}$$

(choose vector $a+bx$ where a, b are any real values)

$$(T \circ S)(a+bx) = a+bx$$

$$a(1)+b(x)$$

$$= a(T \circ S)(1) + b(T \circ S)(x)$$

$$= a(1) + b(x)$$

$$= a+bx \rightarrow \text{Thus, } T \circ S = \text{id}_{P_1}$$

$$[T]_{*y} [S]_{y*} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{array}{l} T \circ S(1) = 1 \\ T \circ S(x) = x \end{array}$$

$$\text{Since } S \circ T = [S]_{y*} [T]_{*y} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ integrating a differentiation loses the constant and thus the } 0 \text{ in the left upper corner does not make } S \circ T = \text{id}_{P_2}.$$