Instructions

1. The statements in Italics are for introducing results and notations that may be used again in this course. You are only required to read and think about them.

2. To receive full credit you must explain how you got your answer.

3. While I encourage collaboration, you must write solutions IN YOUR OWN WORDS. DO NOT SHARE COMPLETE SOLUTIONS before they are due. YOU WILL RECEIVE NO CREDIT if you are found to have copied from whatever source or let others copy your solutions.

4. Workshops must be handwritten (electronic handwriting is allowed) for authentication purposes and submitted on Canvas. Please do NOT include any personal information such as your name and netID in your file. Late homework will NOT be accepted. It is your responsibility to MAKE SURE THAT YOUR SUBMISSIONS ARE SUCCESSFUL AND YOUR FILES ARE LEGIBLE AND COMPLETE. It is also your responsibility that whoever reads your work will understand and enjoy it. Up to 1 point out of 10 may be taken off if your solutions are hard to read or poorly presented.

Workshop 6

1. Let U, V, W be vector spaces and $S : U \to V$, $T : V \to W$ be linear transformations. Define the **composition** $T \circ S : U \to W$ by $T \circ S(u) = T(S(u))$ for all u in U.

a. Show that $T \circ S$ is a linear transformation.

b. Now suppose U is 1-dimensional with basis $\mathfrak{X} = \{u_1\}$, V is 2-dimensional with basis $\mathfrak{Y} = \{v_1, v_2\}$, W is 3-dimensional with basis $\mathfrak{Z} = \{w_1, w_2, w_3\}$. Show $[T \circ S]_{\mathfrak{Z}\mathfrak{X}} = [T]_{\mathfrak{Z}\mathfrak{Y}}[S]_{\mathfrak{Y}\mathfrak{X}}$. *This equality holds for compositions of linear transformations in general, and this is why we defined matrix multiplication the way we did.*

$P_1 \xrightarrow{S} P_2 \xrightarrow{T} P_1$

2. Let $T : \mathcal{P}_2 \to \mathcal{P}_1$ be the linear transformation given by differentiation and $S : \mathcal{P}_1 \to \mathcal{P}_2$ be the linear transformation given by $S(p(x)) = \int_0^x p(t)dt$ for all $p(x)$ in $\mathcal{P}_1$. Let $\mathfrak{X} = \{1, x\}$ and $\mathfrak{Y} = \{1, x, x^2\}$.

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad$ $\mathfrak{X}$ is a basis for P_1

a. Write down the matrices $[T]_{\mathfrak{X}\mathfrak{Y}}$ and $[S]_{\mathfrak{Y}\mathfrak{X}}$.

b. Compute $[T]_{\mathfrak{X}\mathfrak{Y}}[S]_{\mathfrak{Y}\mathfrak{X}}$ and $[S]_{\mathfrak{Y}\mathfrak{X}}[T]_{\mathfrak{X}\mathfrak{Y}}$. Conclude that $T \circ S = id_{\mathcal{P}_1}$. *Note how this is consistent with the fact that differentiation after integration gives back the original function.* Why is $S \circ T$ not $id_{\mathcal{P}_2}$?

$[T\circ S]_{\mathfrak{X}\mathfrak{X}}$ $\quad$ $T\circ S$ $\quad$ $\mathfrak{X}$ $\quad$ $T\circ S : P_1 \to P_1$

3. Let V, W be vector spaces of the same (finite) dimension. A linear transformation $T : V \to W$ is called **invertible** if there exists a linear transformation $S : W \to V$ such that $S \circ T = id_V$ and $T \circ S = id_W$. Show that a linear transformation $T : V \to W$ is invertible if and only if for every basis $\mathfrak{X}$ of V and $\mathfrak{Y}$ of W, $[T]_{\mathfrak{Y}\mathfrak{X}}$ is an invertible matrix.

$S(u_1) = b_{11}v_1 + b_{21}v_2$

$T(v_1) = a_{11}w_1 + a_{21}w_2 + a_{31}w_3$

$T(v_2) = a_{12}w_1 + a_{22}w_2 + a_{32}w_3$

4. Let V, W be vector spaces over $\mathbb{R}/\mathbb{C}$ and $T : V \to W$ be a linear transformation.

a. We define the **kernel** of T (denoted $K(T)$) to be the set of v in V such that $T(v) = 0_W$. Show that $K(T)$ is a subspace of V.

b. We define the **range** of T (denoted $R(T)$) to be the set of w in W such that there exists a v in V with $T(v) = w$. Show that $R(T)$ is a subspace of W.

5. Let T and S be as in problem 2. Find $K(T), R(T), K(S), R(S)$.

$$[T]_{\mathfrak{Z}\mathfrak{Y}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \quad [S]_{\mathfrak{Y}\mathfrak{X}} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}, \quad [T]_{\mathfrak{Z}\mathfrak{Y}}[S]_{\mathfrak{Y}\mathfrak{X}} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} \\ a_{31}b_{11} + a_{32}b_{21} \end{bmatrix}$$

$$T\circ S(u_1) = T(S(u_1)) = T(b_{11}v_1 + b_{21}v_2)$$
$$= b_{11}T(v_1) + b_{21}T(v_2).$$
$$= b_{11}(a_{11}w_1 + a_{21}w_2 + a_{31}w_3)$$

$$+ b_{21}(a_{12}w_1 + a_{22}w_2 + a_{32}w_3)$$

$$[T \circ S]_{\mathcal{Z}\mathcal{X}} = \begin{bmatrix} b_{11}a_{11} + b_{21}a_{12} \\ b_{11}a_{21} + b_{21}a_{22} \\ b_{11}a_{31} + b_{21}a_{32} \end{bmatrix} = (b_{11}a_{11} + b_{21}a_{12})w_1$$

$$+ (b_{11}a_{21} + b_{21}a_{22})w_2$$

$$+ (b_{11}a_{31} + b_{21}a_{32})w_3$$

$\mathcal{Z}$