Workshop 5

$\mathbb{R}^2$

1. Let V be a 2-dimensional vector space with basis $\mathfrak{X} = \{v_1, v_2\}$, write down the matrices $[0]_{\mathfrak{X}\mathfrak{X}}$ and $[id]_{\mathfrak{X}\mathfrak{X}}$.

$T(v_1) = \overset{a_{11}}{0}(v_1) + \overset{a_{21}}{0}(v_2) = 0$

$[0]_{\mathfrak{X}\mathfrak{X}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$T(v_2) = \underset{a_{12}}{0}(v_1) + \underset{a_{22}}{0}(v_2) = 0$

basis $\mathfrak{X} = \{v_1, v_2\}$

Identity matrix $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$v_1 = \overset{a_{11}}{1}(v_1) + \overset{a_{21}}{\cancel{0(v_2)}}$

$v_2 = \underset{a_{12}}{\cancel{0(v_1)}} + \underset{a_{22}}{1}(v_2)$

$[id]_{\mathfrak{X}\mathfrak{X}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. Let U, V, W be vector spaces and $\underline{S : U \to V}$, $\underline{T : V \to W}$ be linear transformations. Define the **composition** $T \circ S : U \to W$ by $T \circ S(u) = T(S(u))$ for all u in U.

a. Show that $T \circ S$ is a linear transformation.

b. Now suppose U is 1-dimensional with basis $\mathfrak{X} = \{u_1\}$, V is 2-dimensional with basis $\mathfrak{Y} = \{v_1, v_2\}$, W is 3-dimensional with basis $\mathfrak{Z} = \{w_1, w_2, w_3\}$. Show $[T \circ S]_{\mathfrak{Z}\mathfrak{X}} = [T]_{\mathfrak{Z}\mathfrak{Y}}[S]_{\mathfrak{Y}\mathfrak{X}}$. *This equality holds for compositions of linear transformations in general, and this is why we defined matrix multiplication the way we did.*

a. $S : U \to V$, $T : V \to W$

$T \circ S : U \to W$ by $T \circ S(u) = T(S(u))$

<u>under addition</u>: for any $\underbrace{u \text{ and } v}_{\text{vectors}}$ in U, $T \circ S(u+v) = T(S(u+v)) = T(S(u)) + T(S(v))$
$= T \circ S(u) + T \circ S(v)$

<u>under scalar multiplication</u>: for any λ in $\mathbb{R}$, $T \circ S(\lambda u) = T(S(\lambda u)) = T(\lambda S(u)) = \lambda(T(S(u))$

$\lambda(T(S(u)) = \lambda T \circ S(u)$

Linear combinations are preserved, so $T \circ S$ is a linear transformation.