Workshop 17

1. The rank k is equal to the number of nonzero eigenvalues. Since symmetric matrices are orthogonally diagonalizable, there exist an orthogonal matrix Q and diagonal matrix D such that $A = QDQ^T$. The rank of a diagonal matrix is equal to the number of nonzero diagonal entries (or number of nonzero eigenvalues). Because Q and Q^T are invertible, both matrices are full rank. Therefore, the rank of matrix A is the same as the rank of D ($\text{rank}(QDQ^T) = \text{rank}(D)$ since Q and Q^T are invertible). Consequently, the rank of all symmetric matrices is equivalent to the number of nonzero entries along its diagonals (or eigenvalues).

2a. $A^TAv = 0$ $\qquad$ $Av = 0$

Plugging in $Av = 0$ into $A^TAv = 0$:

$A^T(0) = 0$

$0 = 0$

Therefore, if $Av = 0$, then $A^TAv = 0$

Multiply v^T on both sides of $A^TAv = 0$:

$v^TA^TAv = 0$

$(Av)^TAv = 0$

$Av \cdot Av = 0$

$\|Av\|^2 = 0$

$\|Av\| = 0$

Because $\|Av\| = 0$, the product Av must be zero.

Therefore, $A^TAv = 0$ only if $Av = 0$.

b. Because $A^TAv = 0$ only if $Av = 0$, we can deduce the Kernel of A^TAv is equal to the Kernel of Av; $K(A^TA) = K(A)$. Therefore, the dimensions of the Kernels are the same. We also Know that the dimension of the domains of A^TA and A are both n, so $\text{rank}(A^TA) = \text{rank}(A)$.