Workshop 16

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d) Since we already proved Q to be an orthogonal matrix that means $QQ^T = I_2$

So we can do the following

$$(Qu) \cdot (Qv) \Rightarrow (Qu)^T (Qv)$$

$$u^T Q^T Q v \Rightarrow u^T v$$

Since a row vector of 1×2 is being multiplied by a column vector of 2×1, you get a 1×1 which is actually equivalent to $u \cdot v$

$$\therefore (Qu) \cdot (Qv) = u \cdot v$$

(2)

a) $v_1 = w_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

$$v_2 = w_2 - \frac{(w_2 \cdot v_1)}{\|v_1\|} \cdot \frac{v_1}{\|v_1\|}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{\sqrt{5}} \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 3/5 \\ 6/5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2/5 \\ -1/5 \end{bmatrix} \rightarrow \begin{bmatrix} \sqrt{30}/5 \\ 2\sqrt{30}/25 \\ -\sqrt{30}/25 \end{bmatrix}$$

$$X = \left\{ \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2\sqrt{5} \end{bmatrix}, \begin{bmatrix} \sqrt{30}/5 \\ 2\sqrt{30}/25 \\ -\sqrt{30}/25 \end{bmatrix} \right\}$$

To get $W^{\perp}$ we find the Kernel of W.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$R_1 \leftarrow R_1 - R_2$ $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$x_1 - x_3 = 0 \rightarrow x_1 = x_3$

$x_2 + 2x_3 = 0 \quad x_2 = -2x_3$

so we get $\begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

Basis of $W^{\perp} = \left\{ \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$

Now to verify $X \cup Y$ is a basis for R^3

$$\left\{ \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} \sqrt{30}/5 \\ 2\sqrt{30}/25 \\ -\sqrt{30}/25 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$$

Using Gaussian elimination (thru calculator)

The matrix can be reduced to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the pivots are 0, that means the vectors are linearly independent and since each element is 3 dimensional $\therefore X \cup Y$ is a basis for $\mathbb{R}^3$