Workshop 15

2.

$$\text{Span}\left\{\begin{bmatrix}0\\1\\0\\1\end{bmatrix},\begin{bmatrix}1\\2\\-1\\0\end{bmatrix},\begin{bmatrix}3\\3\\-3\\-3\end{bmatrix},\begin{bmatrix}-2\\-2\\2\\-2\end{bmatrix}\right\} = \text{Span}\{u_1, u_2, u_3, u_4\}$$

$$\vec{v}_1 = \vec{u}_1$$

$$\vec{v}_1 = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{(\vec{u}_2 \cdot \vec{v}_1)}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$\vec{v}_2 = \begin{bmatrix}1\\2\\-1\\0\end{bmatrix} - \frac{2}{2}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$$

$$\vec{v}_3 = \vec{u}_3 = \frac{(\vec{u}_3 \cdot \vec{v}_1)}{\|v_1\|^2}\vec{v}_1 - \frac{(\vec{u}_3 \cdot \vec{v}_2)}{\|v_2\|^2}\vec{v}_2$$

$$\vec{v}_3 = \begin{bmatrix}3\\3\\-3\\-3\end{bmatrix} - \frac{0}{2}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} - \frac{12}{4}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\vec{v}_4 = \vec{u}_4 - \frac{(\vec{u}_4 \cdot \vec{v}_1)}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{(\vec{u}_4 \cdot \vec{v}_2)}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$\vec{v}_4 = \begin{bmatrix}-2\\-2\\2\\-2\end{bmatrix} - \frac{-4}{2}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} - \frac{-4}{4}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$$

$$= \begin{bmatrix}-2\\-2\\2\\-2\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\\1\end{bmatrix} + 1\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} = \begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}$$

An orthogonal basis is:

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

3a. $y \longrightarrow x$

$$\vec{V_1} = (\vec{V_1} \cdot \vec{u_1})\vec{u_1} + (\vec{V_1} \cdot \vec{u_2})\vec{u_2}$$
$$\vec{V_2} = (\vec{V_2} \cdot \vec{u_1})\vec{u_1} + (\vec{V_2} \cdot \vec{u_2})\vec{u_2}$$

$$[id]_{xy} = \begin{bmatrix} \vec{V_1} \cdot \vec{u_1} & \vec{V_2} \cdot \vec{u_1} \\ \vec{V_1} \cdot \vec{u_2} & \vec{V_2} \cdot \vec{u_2} \end{bmatrix} = Q$$

3b) Show column vectors are orthonormal.

Show the dot product of the two column vectors $= 0$. $(\vec{v_1} \cdot \vec{u_1})(\vec{v_2} \cdot \vec{u_1}) + (\vec{v_1} \cdot \vec{u_2})(\vec{v_2} \cdot \vec{u_2}) = 0$

$$\vec{v_1} \cdot \vec{v_2} = \left[(\vec{v_1} \cdot \vec{u_1})(\vec{u_1}) + (\vec{v_1} \cdot \vec{u_2})(\vec{u_2})\right]\left[(\vec{v_2} \cdot \vec{u_1})(\vec{u_1}) + (\vec{v_2} \cdot \vec{u_2})(\vec{u_2})\right]$$
$$= (\vec{v_1} \cdot \vec{u_1})(\vec{v_2} \cdot \vec{u_1})(\vec{u_1} \cdot \vec{u_1}) + (\vec{v_1} \cdot \vec{u_2})(\vec{v_2} \cdot \vec{u_2})(\vec{u_2} \cdot \vec{u_2})$$
$$\vec{v_1} \cdot \vec{v_2} = (\vec{v_1} \cdot \vec{u_1})(\vec{v_2} \cdot \vec{u_1}) + (\vec{v_1} \cdot \vec{u_2})(\vec{v_2} \cdot \vec{u_2})$$
$$0 = (\vec{v_1} \cdot \vec{u_1})(\vec{v_2} \cdot \vec{u_1}) + (\vec{v_1} \cdot \vec{u_2})(\vec{v_2} \cdot \vec{u_2})$$

we know that $\vec{v_1} \cdot \vec{v_2} = 0$ b/c they are orthogonal

Thus the column vectors are orthogonal.

Show that $(\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_1 \cdot u_1) + (\vec{v}_1 \cdot \vec{u}_2)(\vec{v}_1 \cdot \vec{u}_2) = 1$.

$$\vec{v}_1 \cdot \vec{v}_1 = [(\vec{v}_1 \cdot \vec{u}_1)(\vec{u}_1) + (\vec{v}_1 \cdot \vec{u}_2)(\vec{u}_2)] \cdot [(\vec{v}_1 \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_1 \cdot \vec{u}_2)\vec{u}_2]$$

$$\overset{\|}{1} = (\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_1 \cdot \vec{u}_1) + (\vec{v}_1 \cdot \vec{u}_2)(\vec{v}_1 \cdot \vec{u}_2) \Rightarrow \text{first column vector has norm 1.}$$

Repeat the same process to show that the second column vector also has norm 1.

$$\vec{v}_2 \cdot \vec{v}_2 = [(\vec{v}_2 \cdot \vec{u}_1)(\vec{u}_1) + (\vec{v}_2 \cdot \vec{u}_2)\vec{u}_2][(\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_2 + \vec{u}_2)\vec{u}_2]$$

$$\overset{\|}{1} = (\vec{v}_2 \cdot \vec{u}_1)(\vec{v}_2 \cdot \vec{u}_1) + (\vec{v}_2 \cdot \vec{u}_2)(\vec{v}_2 \cdot \vec{u}_2) \Rightarrow \text{2nd column vector also has norm 1.}$$

Thus the column vectors are orthonormal.

Show row vectors are orthonormal.

$(\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_1 \cdot \vec{u}_2) + (\vec{v}_2 \cdot \vec{u}_1)(\vec{v}_2 \cdot \vec{u}_2) = 0$ must be true.

$$\vec{u}_1 = (\vec{u}_1 \cdot \vec{v}_1)\vec{v}_1 + (\vec{u}_1 \cdot \vec{v}_2)\vec{v}_2$$

$$\vec{u}_2 = (\vec{u}_2 \cdot \vec{v}_1)\vec{v}_1 + (\vec{u}_2 \cdot \vec{v}_2)\vec{v}_2$$

$$\vec{u}_1 \cdot \vec{u}_2 = [(\vec{u}_1 \cdot \vec{v}_1)\vec{v}_1 + (\vec{u}_1 \cdot \vec{v}_2)\vec{v}_2] \cdot [(\vec{u}_2 \cdot \vec{v}_1)\vec{v}_1 + (\vec{u}_2 \cdot \vec{v}_2)\vec{v}_2]$$

$$\| \quad = (\vec{u}_1 \cdot \vec{v}_1)(\vec{u}_2 \cdot \vec{v}_1) + (\vec{u}_1 \cdot \vec{v}_2)(\vec{u}_2 \cdot \vec{v}_2) \quad \text{commutativity}$$

$$0 = (\vec{v}_1 \cdot \vec{u}_1)(\vec{v}_1 \cdot \vec{u}_2) + (\vec{v}_2 \cdot \vec{u}_1)(\vec{v}_2 \cdot \vec{u}_2)$$

Thus the row vectors are orthogonal.

$$\vec{u}_1 \cdot \vec{u}_1 = [(\vec{u}_1 \cdot \vec{v}_1)\vec{v}_1 + (\vec{u}_1 \cdot \vec{v}_2)\vec{v}_2] \cdot [(\vec{u}_1 \cdot \vec{v}_1)\vec{v}_1 + (\vec{u}_1 \cdot \vec{v}_2)\vec{v}_2]$$

$$\overset{\|}{1} = (\vec{u}_1 \cdot \vec{v}_1)(\vec{u}_1 \cdot \vec{v}_1) + (\vec{u}_1 \cdot \vec{v}_2)(\vec{u}_1 \cdot \vec{v}_2) = \text{norm of first row vect.}$$

Thus the first row vector has norm 1. Follow a similar process to show that the second row vector also has norm 1 (use $\vec{u}_2 \cdot \vec{u}_2 = 1$ ... etc).

Therefore, we can conclude that the row vectors in matrix Q are orthonormal.