WORKSHOP 13

1.) $\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 3-\lambda & 2 \\ -1 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

$$2-\lambda\,[(3-\lambda)(1-\lambda)] = 0$$

$$\left.\boxed{\begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{array}}\right\} \text{Eigen values}$$

For $\lambda = 1$:

$$\begin{bmatrix} 2-1 & 0 & 0 \\ 1 & 3-1 & 2 \\ -1 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ +1\cdot(R_3) \\ +1\cdot(R_1) \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \times(1/2) \\ \\ \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \text{RREF}$$

Let $x_3 = 1$

$$\left.\begin{array}{l} x_1 = 0 \\ x_2 + x_3 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = -x_3 \\ x_3 = x_3 \end{array} \Longrightarrow x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\boxed{\text{For } \lambda = 1,\ \text{Eigen vector of } A\ (\vec{v}_1) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$$

eigen vector T =
$0\cdot(1,1,0) + (-1)\cdot(0,0,1) + 1\cdot(1,0,1)$
$= (1,0,0)$

For $\lambda = 2$:

$$\begin{bmatrix} 2-2 & 0 & 0 \\ 1 & 3-2 & 2 \\ -1 & 0 & 1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$= \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \text{Swap } R_1 \text{ and } R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \begin{array}{l} \\ \\ +1\cdot(R_1) \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} -1\cdot(R_3) \\ \\ \\ \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} \leftarrow \text{Swap} \\ \leftarrow R_2 \,\&\, R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \text{RREF}$$

Let $x_3 = 1$

$$\left.\begin{array}{l} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{array} \Rightarrow x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\boxed{\text{For } \lambda = 2, \text{ eigenvector of } A \ (\vec{v}_2) = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}$$

eigenvector of $T =$
$(-1)\cdot(1,1,0) + (-1)\cdot(0,0,1) + 1\cdot(1,0,1) = (0,-1,0)$

For $\lambda = 3$:

$$\begin{bmatrix} 2-3 & 0 & 0 \\ 1 & 3-3 & 2 \\ -1 & 0 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right] \begin{array}{l} \times(-1) \\ \\ \\ \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right] \begin{array}{l} \\ -1\cdot(R_1) \\ +1\cdot(R_1) \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \times(1/2) \\ \times(-1/2) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \\ -1\cdot(R_2) \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_2 = 1$

$$\begin{array}{l} x_1 = 0 \\ x_3 = 0 \\ x_2 = x_2 \end{array} \Rightarrow x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{\text{For } \lambda = 3, \text{ eigenvector of } A \ (\vec{v}_3) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

eigenvector of $T =$
$0\cdot(1,1,0) + 1\cdot(0,0,1) + 0\cdot(1,0,1) =$
$(0,0,1)$.

(2) $T: \mathbb{R}^2 \to \mathbb{R}^2$, the standard basis for $\mathbb{R}^2$ is $\{(1,0),(0,1)\}$

When undergoing the T transformation, the vectors in our standard basis change from $\{(1,0),(0,1)\} \to \{(0,1),(-1,0)\}$

$\therefore$ our A matrix is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

To find eigenvalues we do the following

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \boxed{\lambda^2 + 1} = 0$$

Characteristic polynomial

And the roots are $\pm i$