Workshop 11

1.) a. Compute the determinants of elementary matrices.

1. swap two rows

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = A$$

$$\det(A) = 0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$$

2. Scale a row

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times n \rightarrow \begin{bmatrix} n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B$$

$$\det(B) = n \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 0 \cdot \det \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = n$$

3. add a multiple of a row to another

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \text{row1} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = C$$

$$\det(C) = 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 0 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 1$$

4. swap two columns

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$

$\det(D) = 0\begin{bmatrix}0&0\\0&1\end{bmatrix} - 1\begin{bmatrix}1&0\\0&1\end{bmatrix} + 0\begin{bmatrix}1&0\\0&0\end{bmatrix} = -1$

5. scale a column

$$\begin{array}{c} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \\ \uparrow \\ \times n \end{array} \rightarrow \begin{bmatrix}n&0&0\\0&1&0\\0&0&1\end{bmatrix} = E$$

$\det(E) = n\begin{bmatrix}1&0\\0&1\end{bmatrix} - 0\begin{bmatrix}0&0\\0&1\end{bmatrix} + 0\begin{bmatrix}0&1\\0&0\end{bmatrix} = n$

6. add one column to another

$$\begin{array}{c} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \\ \uparrow \\ +\text{col } 3 \end{array} \rightarrow \begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix} = F$$

$\det(F) = 1\begin{bmatrix}1&0\\0&1\end{bmatrix} - 0\begin{bmatrix}0&0\\1&1\end{bmatrix} + 0\begin{bmatrix}0&1\\1&0\end{bmatrix} = 1$

*For reference, $\det(I_3) = 1$

b. How does performing an elementary row operation on a matrix affect its determinant?

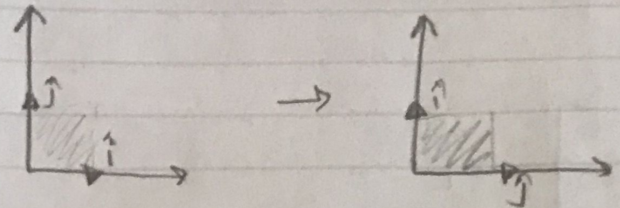
From the work above, we can see a pattern emerge.

① If two rows are swapped, the determinant changes sign.

② If we multiply by a scalar, the determinant gets multiplied by that as well

③ when adding one row/col to another, it doesn't change.

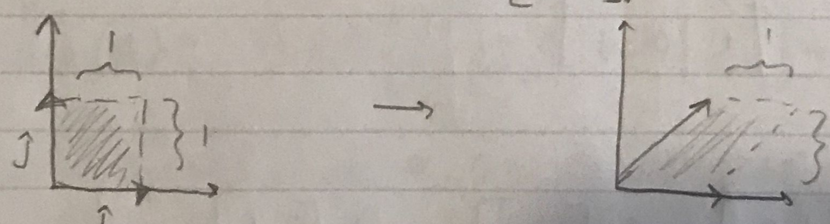
These conclusions intuitively make sense. For 1, we can think of flipping the vectors around.



The determinant becomes smaller as $\hat{i}$ & $\hat{j}$ get closer and it becomes negative when $\hat{i}$ passes $\hat{j}$.

For 2, when we scale a column/row, we are scaling the area by that factor. The determinant measures the factor by which the given space changes.

Finally in 3, adding a column/row would not change the determinant. Consider $\begin{bmatrix}1&0\\0&1\end{bmatrix}$.



When we add the 1st column to the second, we get $\begin{bmatrix}1&1\\0&1\end{bmatrix}$. $\hat{j}$ slides over. However, the area remains unchanged.

2.) How does transposing a matrix affect its determinant

For this problem, we can test an arbitrary matrix to see what will happen.

$\rightarrow$

$a,b,c,d,e,f,g,h,i \in R$

2) For a 2X2 matrix :

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\det(A) = ad - bc$

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$\det(A^T) = ad - bc$

$\therefore \det(A) = \det(A^T)$

For a 3X3 matrix :

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} ; \quad A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$\det(A) = a[ei - fh] - b[di - fg] + c[dh - eg]$$
$$= aei - afh - bdi + bfg + cdh - ceg$$

$$\det(A^T) = a[ei - fh] - d[bi - ch] + g[bf - ec]$$
$$= aei - afh - dbi + chd + gbf - gec$$
$$= aei - afh - bdi + cdh - bfg - ceg$$

$\therefore \det(A) = \det(A^T)$

$\therefore$ For a nxn matrix :

$$\underset{n\times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} ; \quad \underset{n\times n}{A^T} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

$$\det(A) = a_{11} \cdot \det(A_{11}) - a_{21} \cdot \det(A_{21}) + \ldots + (-1)^{n+1} a_{n1} \cdot \det(A_{n1}) \quad \text{—— } ①$$

where A_{ij} is the submatrix of A obtained by removing the i^{th} row and j^{th} column from A

$$\det(A^T) = a_{11} \cdot \det(A^T)_{11} - a_{21} \cdot \det(A^T)_{12} + \ldots + (-1)^{n+1} a_{n1} \cdot \det(A^T)_{1n} \quad \text{—— } ②$$

We can see that $(A^T)_{ij} = (A_{ji})^T$

$\therefore \det(A^T)_{ij} = \det((A_{ji})^T) = \det(A_{ji}) \Rightarrow$ Puting this in equation ②

$$\det(A^T) = a_{11}\cdot\det(A^T)_{11} - a_{21}\cdot\det(A^T)_{12} + \dots + (-1)^{n+1}\cdot a_{n1}\cdot\det(A^T)_{1n}$$

$$= a_{11}\cdot\det(A_{11}) - a_{21}\cdot\det(A_{21}) + \dots + (-1)^{n+1}\cdot a_{n1}\cdot\det(A_{n1})$$

$$= \det(A)$$

$\therefore$ $\boxed{\text{Hence, } \det(A) = \det(A^T) \text{ for any } n\times n \text{ square matrix } A}$