12:20 pm — 3:20 pm, 7/2

1 multiple answer question
10 file upload questions
1 reflection question

part 1 : reflection 30 min

- part 2 & 3 : time window $\Leftrightarrow$ time limit. 6 minutes longer
- part 4 & 5 :

Basis

Def. A basis for a vector space is a set of vectors in this vector space that is
① linearly independent ② span = vector space.

E.g. $\mathbb{R}^3$ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ standard basis

there are many other bases

dimension = number of vectors in a basis

linear transformation $\underset{\mathcal{X}}{T: V} \to \underset{\mathcal{Y}}{W}$ $[T]_{\mathcal{Y}\mathcal{X}}$

orthogonal / orthonormal basis.

compute bases : Gaussian Elimination

Gram - Schmit

Span$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ want to find orthogonal basis

orthonormal

$\vec{v}_1 = \vec{u}_1$

$\vec{v}_1' = \frac{\vec{v}_1}{\|\vec{v}_1\|}$

$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$

$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2' \cdot \vec{v}_1'}{\underbrace{\vec{v}_1' \cdot \vec{v}_1'}_{=1}} \vec{v}_1'$

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Homework 5 Solution

1. Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^m$ are vectors satisfying:

$$\|\boldsymbol{u}\| = 2 \quad \|\boldsymbol{v}\| = 3 \quad \|\boldsymbol{w}\| = 4 \quad \boldsymbol{u}\cdot\boldsymbol{v} = -1 \quad \boldsymbol{u}\cdot\boldsymbol{w} = 2 \quad \boldsymbol{v}\cdot\boldsymbol{w} = -2$$

Compute the following expressions: (2 pts each)

a. $(2\boldsymbol{u}+\boldsymbol{v})\cdot(3\boldsymbol{v}-4\boldsymbol{w})$
b. $\|\boldsymbol{u}+\boldsymbol{v}\|^2$
c. $\|-6\boldsymbol{w}\|$
d. $\|2\boldsymbol{v}-\boldsymbol{w}\|$

a. $(2\boldsymbol{u}+\boldsymbol{v})\cdot(3\boldsymbol{v}-4\boldsymbol{w}) = 6\boldsymbol{u}\cdot\boldsymbol{v} - 8\boldsymbol{u}\cdot\boldsymbol{w} + 3\boldsymbol{v}\cdot\boldsymbol{v} - 4\boldsymbol{v}\cdot\boldsymbol{w}$ (1 pt)$= 6\times(-1) - 8\times 2 + 3\times 3^2 - 4\times(-2) = 13$ (1 pt).

b. $\|\boldsymbol{u}+\boldsymbol{v}\|^2 = (\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{u}\cdot\boldsymbol{u} + 2\boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{v}$ (1 pt)$= 2^2 + 2\times(-1) + 3^2 = 11$ (1 pt).

c. $\|-6\boldsymbol{w}\| = \sqrt{(-6\boldsymbol{w})\cdot(-6\boldsymbol{w})} = 6\sqrt{\boldsymbol{w}\cdot\boldsymbol{w}} = 6\|\boldsymbol{w}\|$ (1 pt)$= 6\times 4 = 24$ (1 pt).

d. $\|2\boldsymbol{v}-\boldsymbol{w}\| = \sqrt{(2\boldsymbol{v}-\boldsymbol{w})\cdot(2\boldsymbol{v}-\boldsymbol{w})} = \sqrt{4\boldsymbol{v}\cdot\boldsymbol{v} - 4\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w}}$ (1 pt)$= \sqrt{4\times 3^2 - 4\times(-2) + 4^2} = 2\sqrt{15}$ (1 pt).

2. Use dot products to represent $\boldsymbol{u} = \begin{bmatrix}3\\1\\2\end{bmatrix}$ as a linear combination of the vectors in the orthogonal set $S = \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\end{bmatrix}\right\}$. (4 pts)

Let $\boldsymbol{v_1} = \begin{bmatrix}1\\0\\1\end{bmatrix}, \boldsymbol{v_2} = \begin{bmatrix}1\\2\\-1\end{bmatrix}, \boldsymbol{v_3} = \begin{bmatrix}1\\-1\\-2\end{bmatrix}$, then

$$\boldsymbol{u} = \frac{\boldsymbol{u}\cdot\boldsymbol{v_1}}{\boldsymbol{v_1}\cdot\boldsymbol{v_1}}\boldsymbol{v_1} + \frac{\boldsymbol{u}\cdot\boldsymbol{v_2}}{\boldsymbol{v_2}\cdot\boldsymbol{v_2}}\boldsymbol{v_2} + \frac{\boldsymbol{u}\cdot\boldsymbol{v_3}}{\boldsymbol{v_3}\cdot\boldsymbol{v_3}}\boldsymbol{v_3}$$

(numerator and denominator of each coefficient is worth 0.5 pt each)

$$= \frac{5}{2}\boldsymbol{v_1} + \frac{3}{6}\boldsymbol{v_2} + \frac{0}{3}\boldsymbol{v_3} = \frac{5}{2}\boldsymbol{v_1} + \frac{1}{2}\boldsymbol{v_2}$$

(1 pt for computation).

3. Find an orthogonal matrix with first column $\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$. (7 pts)

Let $\boldsymbol{u_1} = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}, \boldsymbol{u_2} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \boldsymbol{u_3} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \boldsymbol{u_4} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$, then $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ is linearly independent. We perform Gram-Schmit on this set with normalization. The resulting vectors will form the columns of an orthogonal matrix with the first column being $\boldsymbol{u_1}$.

Columns need to orthonormal

$$\boldsymbol{v_1} = \boldsymbol{u_1} \text{ (already normal).}$$

$$\boldsymbol{v_2} = \boldsymbol{u_2} - (\boldsymbol{u_2}\cdot\boldsymbol{v_1})\boldsymbol{v_1} = \frac{1}{4}\begin{bmatrix}3\\-1\\-1\\-1\end{bmatrix}.\ \boldsymbol{v_2'} = \frac{\boldsymbol{v_2}}{\|\boldsymbol{v_2}\|} = \frac{1}{2\sqrt{3}}\begin{bmatrix}3\\-1\\-1\\-1\end{bmatrix}.$$

$$\boldsymbol{v_3} = \boldsymbol{u_3} - (\boldsymbol{u_3}\cdot\boldsymbol{v_1})\boldsymbol{v_1} - (\boldsymbol{u_3}\cdot\boldsymbol{v_2'})\boldsymbol{v_2'} = \frac{1}{3}\begin{bmatrix}0\\2\\-1\\-1\end{bmatrix}.\ \boldsymbol{v_3'} = \frac{\boldsymbol{v_3}}{\|\boldsymbol{v_3}\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}0\\2\\-1\\-1\end{bmatrix}.$$

$$\boldsymbol{v_4} = \boldsymbol{u_4} - (\boldsymbol{u_4}\cdot\boldsymbol{v_1})\boldsymbol{v_1} - (\boldsymbol{u_4}\cdot\boldsymbol{v_2'})\boldsymbol{v_2'} - (\boldsymbol{u_4}\cdot\boldsymbol{v_3'})\boldsymbol{v_3'} = \frac{1}{2}\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}.\ \boldsymbol{v_4'} = \frac{\boldsymbol{v_4}}{\|\boldsymbol{v_4}\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}.$$

Therefore, $\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0\\ \frac{1}{2} & -\frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{3} & 0\\ \frac{1}{2} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2}\\ \frac{1}{2} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2}\end{bmatrix}$ is one such matrix. (1 pt for finding independent set of vectors; 3 pts for Gram-Schmit; 3 pts for normalization.)

4. When do we have $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$ for vectors $\boldsymbol{u}, \boldsymbol{v}$ in $\mathbb{R}^n$? Explain why. (Hint: Think geometrically.) (4 pts)

Argument 1:

When $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent, $\|\boldsymbol{u}+\boldsymbol{v}\| < \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$, because the length of one side of a triangle is less than the sum of the lengths of the other two sides. (2 pts)

When $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly dependent, we have $\boldsymbol{u} = a\boldsymbol{v}$ for some a in $\mathbb{R}$ or $\boldsymbol{v} = b\boldsymbol{u}$ for some b in $\mathbb{R}$. Suppose $\boldsymbol{u} = a\boldsymbol{v}$ for some a in $\mathbb{R}$. Then $\|\boldsymbol{u}+\boldsymbol{v}\| = \|a\boldsymbol{v}+\boldsymbol{v}\| = |a+1|\|\boldsymbol{v}\|$, $\|\boldsymbol{u}\|+\|\boldsymbol{v}\| = \|a\boldsymbol{u}\|+\|\boldsymbol{v}\| = (|a|+1)\|\boldsymbol{v}\|$. Therefore, when $|a+1| = |a|+1$ or $\|\boldsymbol{v}\|$, that is, when $a \geq 0$ or $\boldsymbol{v} = \mathbf{0}$, we have $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$. Similarly, when $\boldsymbol{u} = \mathbf{0}$ or $\boldsymbol{v} = b\boldsymbol{u}$ for some non-negative b in $\mathbb{R}$, we have $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$. In summary, $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$ exactly when $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly dependent (1 pt) and $\boldsymbol{u}\cdot\boldsymbol{v} \geq 0$ (1 pt).

Argument 2:

Since both sides are non-negative, $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$ is equivalent to $\|\boldsymbol{u}+\boldsymbol{v}\|^2 = (\|\boldsymbol{u}\|+\|\boldsymbol{v}\|)^2$ (1 pt), or $(\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{u}\cdot\boldsymbol{u}+2\|\boldsymbol{u}\|\|\boldsymbol{v}\|+\boldsymbol{v}\cdot\boldsymbol{v}$, i.e., $\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|$. *Note that Cauchy-Schwartz inequality says* $\boldsymbol{u}\cdot\boldsymbol{v} \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$. When $\boldsymbol{u} = \mathbf{0}$ or $\boldsymbol{v} = \mathbf{0}$, we have $\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\| = 0$; when $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero, $\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos\theta$, where θ is the angle from $\boldsymbol{u}$ to $\boldsymbol{v}$ (1 pt). Therefore, the equality holds exactly when $\boldsymbol{u} = \mathbf{0}$ or $\boldsymbol{v} = \mathbf{0}$ or $\cos\theta = 1$ (2 pts).

5. Finish Workshop 16 Problem 2b. (5 pts)

In part a we found $\boldsymbol{w_1} = \frac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\2\end{bmatrix}$, $\boldsymbol{w_2} = \frac{1}{\sqrt{30}}\begin{bmatrix}5\\2\\-1\end{bmatrix}$, $\boldsymbol{z_1} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\1\end{bmatrix}$ and that $\mathfrak{X} = \{\boldsymbol{w_1}, \boldsymbol{w_2}\}$ and $\mathfrak{Y} = \{\boldsymbol{z_1}\}$ are orthonormal bases for W and $W^\perp$, respectively. Let $\boldsymbol{w} = (\boldsymbol{b}\cdot\boldsymbol{w_1})\boldsymbol{w_1} + (\boldsymbol{b}\cdot\boldsymbol{w_2})\boldsymbol{w_2}$ (2 pts)$= \begin{bmatrix}5/6\\7/3\\23/6\end{bmatrix}$ (1 pt), and $\boldsymbol{z} = (\boldsymbol{b}\cdot\boldsymbol{z_1})\boldsymbol{z_1}$ (1 pt) $= \frac{1}{6}\begin{bmatrix}1\\-2\\1\end{bmatrix}$ (1 pt), then $\mathbf{w} \in W$, $\mathbf{z} \in W^\perp$ and $\mathbf{b} = \mathbf{w} + \mathbf{z}$.

6. Let $\boldsymbol{u} = \begin{bmatrix}1\\4\\-1\end{bmatrix}$ and $S = \left\{\frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}$.

a. Check that S is orthonormal. (4 pts)

Let $\boldsymbol{v_1} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\1\end{bmatrix}$, $\boldsymbol{v_2} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}$. Check $\boldsymbol{v_1}\cdot\boldsymbol{v_2} = 0$ (2 pts), $\boldsymbol{v_1}\cdot\boldsymbol{v_1} = 1$ (1 pt), $\boldsymbol{v_2}\cdot\boldsymbol{v_2} = 1$ (1 pt) (details omitted).

b. Find the vector $\boldsymbol{w}$ in the span of S that is closest to $\boldsymbol{u}$. (4 pts)

$\boldsymbol{w} = (\boldsymbol{u} \cdot \boldsymbol{v_1})\boldsymbol{v_1} + (\boldsymbol{u} \cdot \boldsymbol{v_2})\boldsymbol{v_2} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$. (2 pts for correct formula; 2 pts for computation.)

c. Find the distance between $\boldsymbol{w}$ and $\boldsymbol{u}$. (4 pts)

The distance $\|\boldsymbol{w} - \boldsymbol{u}\| = 0$ (2 pts) since $\boldsymbol{w} = \boldsymbol{u}$ (2 pts). ($\boldsymbol{u}$ *is in the span of* S.)

linear transformation $T: V \to W$ $\quad A = [T]_{\gamma\beta}$

$K(T) = \{ v \in V \mid T(v) = \underset{\text{vector}}{0_W} \}$ subspace of V.

$R(T) = \{ w \in W$ such that there exists $v \in V$ w/ $T(v) = w \}$ subspace of W.

$\dim K(T) + \dim R(T) = \dim V$. # columns in A $\quad T: \mathbb{R}^3 \to \mathbb{R}^2$

$\dim K(T)$ = nullity of T

$\dim R(T)$ = rank of T = rank of A

$T: \mathbb{R}^{100} \to \mathbb{R}^{400}$

$$[T]_{\gamma\beta} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 2 & \cdots & 1 \end{bmatrix} = A$$

$K(T)$: "$A\vec{x} = \vec{0}$"

$\dim K(T)$

$R(T)$: "column span of A"

RREF of $\underline{A}$: $\quad$ rank = # of pivots

nullity + rank = # of columns

Elementary row & column operations don't change the rank.

invertible

$\operatorname{rank}(PAQ) = \operatorname{rank}(A)$.

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diagonalizable

Workshop 17

1. Let A be an $n \times n$ symmetric matrix of rank k. What can you say about its eigenvalues?

2. Let A be an $m \times n$ matrix.

a. Let $\boldsymbol{v}$ be an $n \times 1$ column vector. Prove that $A^T A\boldsymbol{v} = \boldsymbol{0}$ if and only if $A\boldsymbol{v} = \boldsymbol{0}$. (Hint: dot product may be helpful.)

b. Use part A to show $A^T A$ and A have the same rank.

3. *An $n \times n$ matrix A is said to be* ***positive definite*** *if A is symmetric and $\boldsymbol{v}^T A\boldsymbol{v} > 0$ for every nonzero column vector $\boldsymbol{v}$ in $\mathbb{R}^n$; it is said to be* ***positive semidefinite*** *if A is symmetric and $\boldsymbol{v}^T A\boldsymbol{v} \geq 0$ for every column vector $\boldsymbol{v}$ in $\mathbb{R}^n$.* Let B be a symmetric matrix.

a. Prove that B is positive definite if and only if all of its eigenvalues are positive.

b. State and prove a characterization of positive semidefinite matrices analogous to that in part a.

$$A = QDQ^T$$

$$\text{rank}(A) = \text{rank}(QDQ^{-1}) = \text{rank}(D)$$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \xrightarrow{GE} \begin{bmatrix} 1 & & & 0 \\ 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}$$

rank = # of nonzero diagonal entries

# of nonzero eigenvalues

(counted w/ multi.)

$$A = QDQ^{-1}$$

Orthogonal Complement & Projection.

$\perp \neq T$

$W = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ subspace of V

$\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}$

$\mathbb{R}^3$

$W^\perp = \{\vec{v}$ in $\mathbb{R}^3$ that is orthogonal to $W\}$

subspace of V.

$\vec{v} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = 0, \quad \vec{v} \cdot \begin{bmatrix}1\\0\\-1\end{bmatrix} = 0$

$\Updownarrow$

$\begin{bmatrix}1 & 2 & 3\\ 1 & 0 & -1\end{bmatrix} \vec{v} = \vec{0}$

vector

$W \cap W^\perp = \{0\}$

basis for W $\sqcup$ basis for $W^\perp$ = basis for V

$$\underbrace{\dim W + \dim W^\perp = \dim V}$$

3 0 3

$\{0\}$

orthogonal basis

x $\qquad$ y. $\qquad$ $x \cup y$

$\vec{u} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$

$\vec{u} \cdot \vec{v}_1 = (c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) \cdot \vec{v}_1$

$= c_1 (\vec{v}_1 \cdot \vec{v}_1) = 1$

x $\qquad$ y

$\{\vec{v}_1, \dots, \vec{v}_n\}$ orthonormal basis for V.

$\vec{u}$ in V, $\vec{u} = \underbrace{(\vec{u}\cdot\vec{v}_1)\vec{v}_1} + \dots + \underbrace{(\vec{u}\cdot\vec{v}_n)\vec{v}_n}$ ✱.

if just orthogonal, $\vec{u} = \dfrac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \dots + \dfrac{\vec{u}\cdot\vec{v}_n}{\vec{v}_n\cdot\vec{v}_n}\vec{v}_n$

if just basis: $\vec{u} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$

$W \quad W^{\perp}$

$x \sqcup y \to$ basis for V

orthonormal

Orthogonal decomposition theorem: $\vec{v} \in V \supseteq W$ subspace.

there exist unique $\vec{w} \in W$, $\vec{z} \in W^{\perp}$ such $\vec{v} = \vec{w} + \vec{z}$

inconsistent system $A\vec{x} = \vec{b}$

||

no solution

W = column span of A

solve $A\vec{x} = U_W(\vec{b})$ instead

$\|A\vec{x}_0 - \vec{b}\|^2 = \|U_W(\vec{b}) - \vec{b}\|^2$

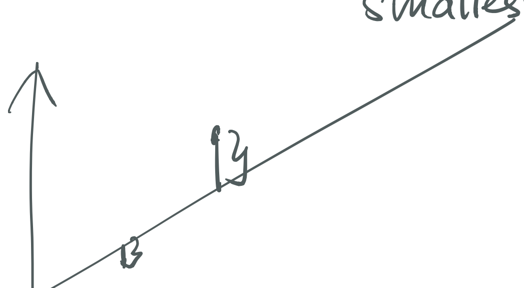
smallest

square

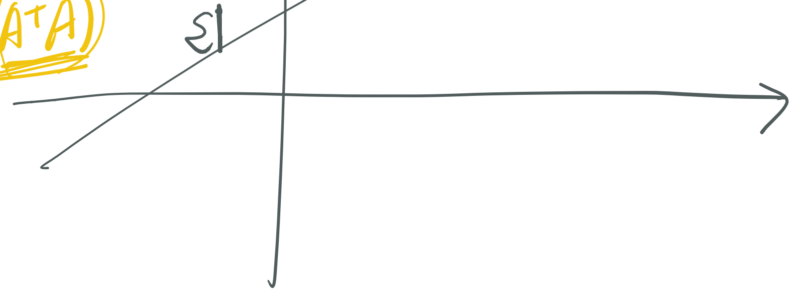
↑

$\boxed{A^TA}\vec{x} = A^T\vec{b}$

unique solu?



rank(A) = rank(A^TA)



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Homework 5 Solution

1. Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^m$ are vectors satisfying:

$$\|\boldsymbol{u}\| = 2 \quad \|\boldsymbol{v}\| = 3 \quad \|\boldsymbol{w}\| = 4 \quad \boldsymbol{u}\cdot\boldsymbol{v} = -1 \quad \boldsymbol{u}\cdot\boldsymbol{w} = 2 \quad \boldsymbol{v}\cdot\boldsymbol{w} = -2$$

Compute the following expressions: (2 pts each)

a. $(2\boldsymbol{u}+\boldsymbol{v})\cdot(3\boldsymbol{v}-4\boldsymbol{w})$
b. $\|\boldsymbol{u}+\boldsymbol{v}\|^2$
c. $\|-6\boldsymbol{w}\|$
d. $\|2\boldsymbol{v}-\boldsymbol{w}\|$

a. $(2\boldsymbol{u}+\boldsymbol{v})\cdot(3\boldsymbol{v}-4\boldsymbol{w}) = 6\boldsymbol{u}\cdot\boldsymbol{v} - 8\boldsymbol{u}\cdot\boldsymbol{w} + 3\boldsymbol{v}\cdot\boldsymbol{v} - 4\boldsymbol{v}\cdot\boldsymbol{w}$ (1 pt)$= 6\times(-1) - 8\times 2 + 3\times 3^2 - 4\times(-2) = 13$ (1 pt).
b. $\|\boldsymbol{u}+\boldsymbol{v}\|^2 = (\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{u}\cdot\boldsymbol{u}+2\boldsymbol{u}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{v}$ (1 pt)$= 2^2+2\times(-1)+3^2 = 11$ (1 pt).
c. $\|-6\boldsymbol{w}\| = \sqrt{(-6\boldsymbol{w})\cdot(-6\boldsymbol{w})} = 6\sqrt{\boldsymbol{w}\cdot\boldsymbol{w}} = 6\|\boldsymbol{w}\|$ (1 pt)$= 6\times 4 = 24$ (1 pt).
d. $\|2\boldsymbol{v}-\boldsymbol{w}\| = \sqrt{(2\boldsymbol{v}-\boldsymbol{w})\cdot(2\boldsymbol{v}-\boldsymbol{w})} = \sqrt{4\boldsymbol{v}\cdot\boldsymbol{v} - 4\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w}}$ (1 pt)$= \sqrt{4\times 3^2 - 4\times(-2) + 4^2} = 2\sqrt{15}$ (1 pt).

2. Use dot products to represent $\boldsymbol{u} = \begin{bmatrix}3\\1\\2\end{bmatrix}$ as a linear combination of the vectors in the orthogonal set $S = \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\end{bmatrix}\right\}$. (4 pts)

Let $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{v_2} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \boldsymbol{v_3} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, then

$$\boldsymbol{u} = \frac{\boldsymbol{u} \cdot \boldsymbol{v_1}}{\boldsymbol{v_1} \cdot \boldsymbol{v_1}}\boldsymbol{v_1} + \frac{\boldsymbol{u} \cdot \boldsymbol{v_2}}{\boldsymbol{v_2} \cdot \boldsymbol{v_2}}\boldsymbol{v_2} + \frac{\boldsymbol{u} \cdot \boldsymbol{v_3}}{\boldsymbol{v_3} \cdot \boldsymbol{v_3}}\boldsymbol{v_3}$$

(numerator and denominator of each coefficient is worth 0.5 pt each)

$$= \frac{5}{2}\boldsymbol{v_1} + \frac{3}{6}\boldsymbol{v_2} + \frac{0}{3}\boldsymbol{v_3} = \frac{5}{2}\boldsymbol{v_1} + \frac{1}{2}\boldsymbol{v_2}$$

(1 pt for computation).

3. Find an orthogonal matrix with first column $\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$. (7 pts)

Let $\boldsymbol{u_1} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \boldsymbol{u_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{u_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{u_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, then $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ is linearly independent. We perform Gram-Schmit on this set with normalization. The resulting vectors will form the columns of an orthogonal matrix with the first column being $\boldsymbol{u_1}$.

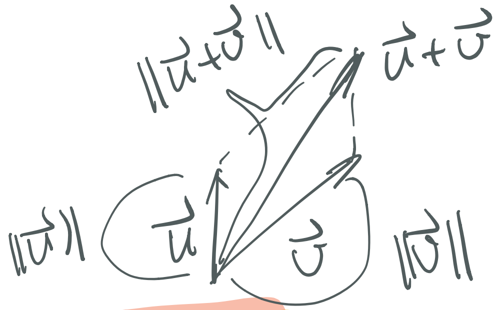
$$\boldsymbol{v_1} = \boldsymbol{u_1} \text{ (already normal).}$$

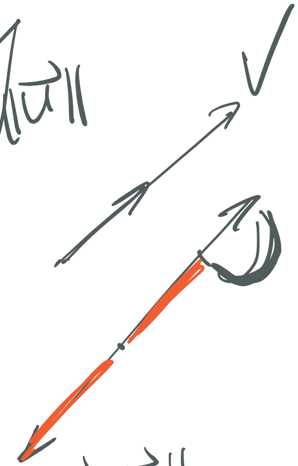
$$\boldsymbol{v_2} = \boldsymbol{u_2} - (\boldsymbol{u_2} \cdot \boldsymbol{v_1})\boldsymbol{v_1} = \frac{1}{4}\begin{bmatrix} 3 \\ -1 \\ -1 \\ -1 \end{bmatrix}. \ \boldsymbol{v_2'} = \frac{\boldsymbol{v_2}}{\|\boldsymbol{v_2}\|} = \frac{1}{2\sqrt{3}}\begin{bmatrix} 3 \\ -1 \\ -1 \\ -1 \end{bmatrix}.$$

$$\boldsymbol{v_3} = \boldsymbol{u_3} - (\boldsymbol{u_3} \cdot \boldsymbol{v_1})\boldsymbol{v_1} - (\boldsymbol{u_3} \cdot \boldsymbol{v_2'})\boldsymbol{v_2'} = \frac{1}{3}\begin{bmatrix} 0 \\ 2 \\ -1 \\ -1 \end{bmatrix}. \ \boldsymbol{v_3'} = \frac{\boldsymbol{v_3}}{\|\boldsymbol{v_3}\|} = \frac{1}{\sqrt{6}}\begin{bmatrix} 0 \\ 2 \\ -1 \\ -1 \end{bmatrix}.$$

$$\boldsymbol{v_4} = \boldsymbol{u_4} - (\boldsymbol{u_4} \cdot \boldsymbol{v_1})\boldsymbol{v_1} - (\boldsymbol{u_4} \cdot \boldsymbol{v_2'})\boldsymbol{v_2'} - (\boldsymbol{u_4} \cdot \boldsymbol{v_3'})\boldsymbol{v_3'} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}. \ \boldsymbol{v_4'} = \frac{\boldsymbol{v_4}}{\|\boldsymbol{v_4}\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Therefore, $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{3} & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} \end{bmatrix}$ is one such matrix. (1 pt for finding independent set of vectors; 3 pts for Gram-Schmit; 3 pts for normalization.)





$\|\vec{u}+\vec{v}\|$
$\|$
$\|\vec{u}\|+\|\vec{v}\|$

4. When do we have $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$ for vectors $\boldsymbol{u}, \boldsymbol{v}$ in $\mathbb{R}^n$? Explain why. (Hint: Think geometrically.) (4 pts)

Argument 1:

When $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent, $\|\boldsymbol{u}+\boldsymbol{v}\| < \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$, because the length of one side of a triangle is less than the sum of the lengths of the other two sides. (2 pts)

When $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly dependent, we have $\boldsymbol{u} = a\boldsymbol{v}$ for some a in $\mathbb{R}$ or $\boldsymbol{v} = b\boldsymbol{u}$ for some b in $\mathbb{R}$. Suppose $\boldsymbol{u} = a\boldsymbol{v}$ for some a in $\mathbb{R}$. Then $\|\boldsymbol{u}+\boldsymbol{v}\| = \|a\boldsymbol{v}+\boldsymbol{v}\| = |a+1|\|\boldsymbol{v}\|$, $\|\boldsymbol{u}\|+\|\boldsymbol{v}\| = \|a\boldsymbol{u}\|+\|\boldsymbol{v}\| = (|a|+1)\|\boldsymbol{v}\|$. Therefore, when $|a+1| = |a|+1$ or $\|\boldsymbol{v}\|$, that is, when $a \geq 0$ or $\boldsymbol{v} = \boldsymbol{0}$, we have $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$. Similarly, when $\boldsymbol{u} = \boldsymbol{0}$ or $\boldsymbol{v} = b\boldsymbol{u}$ for some non-negative b in $\mathbb{R}$, we have $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$. In summary, $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$ exactly when $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly dependent (1 pt) and $\boldsymbol{u}\cdot\boldsymbol{v} \geq 0$ (1 pt).

Argument 2:

Since both sides are non-negative, $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$ is equivalent to $\|\boldsymbol{u}+\boldsymbol{v}\|^2 = (\|\boldsymbol{u}\|+\|\boldsymbol{v}\|)^2$ (1 pt), or $(\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{u}\cdot\boldsymbol{u}+2\|\boldsymbol{u}\|\|\boldsymbol{v}\|+\boldsymbol{v}\cdot\boldsymbol{v}$, i.e., $\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|$. *Note that Cauchy-Schwartz inequality says* $\boldsymbol{u}\cdot\boldsymbol{v} \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$. When $\boldsymbol{u} = \boldsymbol{0}$ or $\boldsymbol{v} = \boldsymbol{0}$, we have $\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\| = 0$; when $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero, $\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos\theta$, where θ is the angle from $\boldsymbol{u}$ to $\boldsymbol{v}$ (1 pt). Therefore, the equality holds exactly when $\boldsymbol{u} = \boldsymbol{0}$ or $\boldsymbol{v} = \boldsymbol{0}$ or $\cos\theta = 1$ (2 pts).

$\mathfrak{X} \cup \mathfrak{Y} = \{\vec{w}_1, \vec{w}_2, \vec{z}_1\}$
orthonormal basis for $\mathbb{R}^3$

5. Finish Workshop 16 Problem 2b. (5 pts)

$\vec{b} = (\vec{b}\cdot\vec{w}_1)\vec{w}_1 + (\vec{b}\cdot\vec{w}_2)\vec{w}_2 + (\vec{b}\cdot\vec{z}_1)\vec{z}_1$

In part a we found $\boldsymbol{w_1} = \frac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\2\end{bmatrix}$, $\boldsymbol{w_2} = \frac{1}{\sqrt{30}}\begin{bmatrix}5\\2\\-1\end{bmatrix}$, $\boldsymbol{z_1} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\1\end{bmatrix}$ and that $\mathfrak{X} = \{\boldsymbol{w_1}, \boldsymbol{w_2}\}$ and $\mathfrak{Y} = \{\boldsymbol{z_1}\}$ are orthonormal bases for W and $W^\perp$, respectively.

Let $\boldsymbol{w} = (\boldsymbol{b}\cdot\boldsymbol{w_1})\boldsymbol{w_1} + (\boldsymbol{b}\cdot\boldsymbol{w_2})\boldsymbol{w_2}$ (2 pts)$= \begin{bmatrix}5/6\\7/3\\23/6\end{bmatrix}$ (1 pt), and $\boldsymbol{z} = (\boldsymbol{b}\cdot\boldsymbol{z_1})\boldsymbol{z_1}$ (1 pt) $= \frac{1}{6}\begin{bmatrix}1\\-2\\1\end{bmatrix}$ (1 pt), then $\mathbf{w} \in W$, $\mathbf{z} \in W^\perp$ and $\mathbf{b} = \mathbf{w}+\mathbf{z}$.

6. Let $\boldsymbol{u} = \begin{bmatrix}1\\4\\-1\end{bmatrix}$ and $S = \left\{\frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}$.

a. Check that S is orthonormal. (4 pts)

Let $\boldsymbol{v_1} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\1\end{bmatrix}$, $\boldsymbol{v_2} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}$. Check $\boldsymbol{v_1}\cdot\boldsymbol{v_2} = 0$ (2 pts), $\boldsymbol{v_1}\cdot\boldsymbol{v_1} = 1$ (1 pt), $\boldsymbol{v_2}\cdot\boldsymbol{v_2} = 1$ (1 pt) (details omitted).

b. Find the vector $\boldsymbol{w}$ in the span of S that is closest to $\boldsymbol{u}$. (4 pts)

$\boldsymbol{w} = (\boldsymbol{u} \cdot \boldsymbol{v_1})\boldsymbol{v_1} + (\boldsymbol{u} \cdot \boldsymbol{v_2})\boldsymbol{v_2} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$. (2 pts for correct formula; 2 pts for computation.)

c. Find the distance between $\boldsymbol{w}$ and $\boldsymbol{u}$. (4 pts)

The distance $\|\boldsymbol{w} - \boldsymbol{u}\| = 0$ (2 pts) since $\boldsymbol{w} = \boldsymbol{u}$ (2 pts). ($\boldsymbol{u}$ *is in the span of* S.)

$A = QBQ^{-1}$

similar

$B = Q^{-1}AQ$

$\det(A) = \det(QBQ^{-1})$

$= \det(Q)\det(B)\det(Q^{-1})$

$= \det(\underbrace{QQ^{-1}}_{I})\det(B)$

$= \det(B).$

$B = PCP^{-1}$

$A = QPCP^{-1}Q^{-1}$

$= (QP)C(QP)^{-1}$