Announcements

1. Final exam will be THURSDAY 7/2 !

2. Tomorrow's class will be spent answering questions. Let me know what you'd like me to talk about by filling in the Google form.
3. I will post a few more references about applications on our course page this weekend.

$$\vec{v}^T(A^TA)\vec{v} = (\vec{v}^TA^T)(A\vec{v}) = (A\vec{v})\cdot(A\vec{v}) = \|A\vec{v}\|^2 \geq 0$$

$$A^TA = (A^TA)^T = A^T(A^T)^T$$

Related: Singular Value Decomposition (SVD)

$n \times n$.

Let A be an $m \times n$ matrix of rank k, then A^TA is an $n \times n$ symmetric positive semidefinite matrix with nonnegative eigenvalues (rank k)

$\lambda_1 \geq \cdots \geq \lambda_k > 0 = \underbrace{\cdots}_{n-k} = 0$. Let $\sigma_i = \sqrt{\lambda_i}$, $i = 1, \ldots, k$.

(These are called singular values of A) and

$$\Sigma = \underset{m \times n}{\left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_k & \\ \hline & 0 & & 0 \end{array}\right]},$$

then there exist an $m \times m$ orthogonal matrix U and $n \times n$ orthogonal matrix V such that $A = U\Sigma V^T$. (This is called the singular value decomposition of A.)

A^T I

Note: We then have

$$A^TA = V\Sigma^T U^T U \Sigma V^T$$
$$\underset{n\times n}{} = V(\Sigma^T\Sigma)V^T$$
$$= V \underbrace{D_1}_{n\times n} V^T \quad ①$$

$$\Sigma^T\Sigma = \begin{bmatrix} \sigma_1^2 & & & & \\ & \ddots & & & 0 \\ & & \sigma_k^2 & & \\ 0 & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}_{n\times n}$$

and

$$AA^T = U\Sigma V^T V \Sigma^T U^T$$
$$\underset{m\times m}{} = U(\Sigma\Sigma^T)U^T$$
$$= U \underbrace{D_2}_{m\times m} U^T \quad ②$$

① and ② are orthogonal diagonalizations of the symmetric matrices A^TA and AA^T, respectively.

Example of Application: Image Compression

A grayscale image is represented by an $m\times n$ matrix

$$A = \underset{m\times m}{U}\Sigma \underset{n\times n}{V^T} = [\vec{u}_1 | \cdots | \vec{u}_m] \left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_k & \\ \hline & 0 & & 0 \end{array}\right] \begin{bmatrix} \vec{v}_1^{\,T} \\ \vdots \\ \vec{v}_n^{\,T} \end{bmatrix}$$

$$= \vec{u}_1\sigma_1\vec{v}_1^{\,T} + \cdots + \vec{u}_k\sigma_k\vec{v}_k^{\,T} \quad (\sigma_1 \geq \cdots \geq \sigma_k > 0) \quad ①$$

Storage: $\quad m \quad 1 \quad n$

A —— mn numbers

The first l terms in ① —— $(m+n+1)l$.

When l is significantly smaller than $\frac{n}{2}$, replacing A with the data of the first l terms in ① results in image compression.

Instructions

1. The statements in Italics are for introducing results and notations that may be used again in this course. You are only required to read and think about them.

2. To receive full credit you must explain how you got your answer.

3. While I encourage collaboration, you must write solutions IN YOUR OWN WORDS. DO NOT SHARE COMPLETE SOLUTIONS before they are due. YOU WILL RECEIVE NO CREDIT if you are found to have copied from whatever source or let others copy your solutions.

4. Workshops must be handwritten (electronic handwriting is allowed) for authentication purposes and submitted on Canvas. Please do NOT include any personal information such as your name and netID in your file. Late homework will NOT be accepted. It is your responsibility to MAKE SURE THAT YOUR SUBMISSIONS ARE SUCCESSFUL AND YOUR FILES ARE LEGIBLE AND COMPLETE. It is also your responsibility that whoever reads your work will understand and enjoy it. Up to 1 point out of 10 may be taken off if your solutions are hard to read or poorly presented.

Workshop 17

1. Let A be an $n \times n$ symmetric matrix of rank k. What can you say about its eigenvalues?

2. Let A be an $m \times n$ matrix.

a. Let $\boldsymbol{v}$ be an $n \times 1$ column vector. Prove that $A^T A\boldsymbol{v} = \mathbf{0}$ if and only if $A\boldsymbol{v} = \mathbf{0}$. (Hint: dot product may be helpful.)

b. Use part A to show $A^T A$ and A have the same rank.

3. *An $n \times n$ matrix A is said to be **positive definite** if A is symmetric and $\mathbf{v}^T A\mathbf{v} > 0$ for every nonzero column vector $\boldsymbol{v}$ in $\mathbb{R}^n$; it is said to be **positive semidefinite** if A is symmetric and $\mathbf{v}^T A\mathbf{v} \geq 0$ for every column vector $\boldsymbol{v}$ in $\mathbb{R}^n$.* Let B be a symmetric matrix.

a. Prove that B is positive definite if and only if all of its eigenvalues are positive.

b. State and prove a characterization of positive semidefinite matrices analogous to that in part a.

B is positive semidefinite if and only all of its eigenvalues are nonnegative.

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