Def. An orthogonal matrix is a change of basis matrix between orthonormal bases: if $\mathcal{X}, \mathcal{Y}$ are orthonormal bases for a subspace V of $\mathbb{R}^n$, then $[id_V]_{\mathcal{X}\mathcal{Y}}$ is an orthogonal matrix.

$k \times k$

$k = \dim V$

E.g. 1×1 orthogonal matrix? $1, -1$

orthonormal basis for $\mathbb{R}$: $\{1\}, \{-1\}$

Note: If P, Q are $n \times n$ orthogonal matrices, then so are PQ and P^{-1}.

$$\underbrace{[id_V]_{\mathcal{X}\mathcal{Y}}}_{P} \underbrace{[id_V]_{\mathcal{Y}\mathcal{Z}}}_{Q} = [id_V]_{\mathcal{X}\mathcal{Z}}$$

$$[id_V]_{\mathcal{X}\mathcal{Y}}^{-1} = [id_V]_{\mathcal{Y}\mathcal{X}}$$

Theorem. Q is an $n \times n$ orthogonal matrix (1)

$\iff$ Columns of Q form an orthonormal set of vectors (2)

$\iff$ $Q^TQ = I_n$ (3) $Q^T\cancel{Q \cdot Q^{-1}} = I_n Q^{-1}$

$\iff$ Q is invertible and $Q^{-1} = Q^T$ (4)

$\iff$ $QQ^T = I_n$ (5)

$$\det(Q^TQ) = \det(I_n)$$
$$= \underline{\det(Q^T)}\,\underline{\det(Q)} = 1$$
$$\det^2(Q)$$

$\iff$ Rows of Q for an orthonormal set of vectors (6)

$\iff$ $(Q\vec{u}) \cdot (Q\vec{v}) = \vec{u} \cdot \vec{v}$ for any column vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ (i.e., Q preserves dot products) (7)

$\iff$ $\|Q\vec{u}\| = \|\vec{u}\|$ for any column vector $\vec{u}$ in $\mathbb{R}^n$ (i.e., Q preserves norms) (8)

(8) $\Rightarrow$ (2)

$$\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = Q$$

Proof: (1) $\iff$ (2) $\iff$ (3)

(1), (3) $\Rightarrow$ (4)

(4) $\Rightarrow$ (5) $\iff$ (6)

(4) $\iff$ $QQ^T = Q^TQ = I$

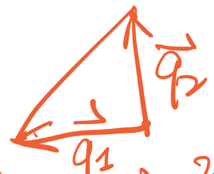
(4) $\Rightarrow$ (7) $\Rightarrow$ (8)

$\vec{e}_1$ $\vec{q}_1$

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}q_{11}\\q_{21}\end{bmatrix} \qquad \|Q(\vec{e}_1+\vec{e}_2)\| = \|\vec{q}_1+\vec{q}_2\|$$

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}q_{12}\\q_{22}\end{bmatrix} \qquad \| \quad \|\vec{e}_1+\vec{e}_2\|$$

$$\qquad\quad \vec{e}_2 \qquad\quad \vec{q}_2$$



$$\| \quad 2 = \|\vec{q}_1\|^2 + \|\vec{q}_2\|^2$$

$$Q\vec{e}_1 = \vec{q}_1, \quad Q\vec{e}_2 = \vec{q}_2 \qquad \|Q\vec{e}_1\|^2 = \|\vec{e}_1\|^2 = 1$$

$$\overset{A}{\begin{bmatrix}\vec{a}_1^T\\ \vdots \\ \vec{a}_m^T\end{bmatrix}}\vec{x} = \begin{bmatrix}\vec{a}_1^T\vec{x}\\ \vdots \\ \vec{a}_m^T\vec{x}\end{bmatrix} = \begin{bmatrix}\vec{a}_1\cdot\vec{x}\\ \vdots \\ \vec{a}_m\cdot\vec{x}\end{bmatrix} = \begin{bmatrix}0\\ \vdots \\ 0\end{bmatrix} \qquad \begin{matrix}\| \\ \|\vec{q}_1\|^2 \\ \|\vec{q}_2\| = 1\end{matrix}$$

Corollary. If Q is an orthogonal matrix, then $\det(Q) = \pm 1$

matrix multiplication

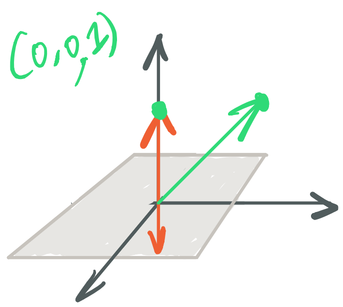
$$\begin{bmatrix}\vec{a}_1^T\\ \vec{a}_2^T\\ \vdots \\ \vec{a}_m^T\end{bmatrix}\times\vec{b} = \begin{bmatrix}\vec{a}_1\cdot\vec{b}\\ \vec{a}_2\cdot\vec{b}\\ \vdots \\ \vec{a}_m\cdot\vec{b}\end{bmatrix} = \begin{bmatrix}\vec{a}_1^T\vec{b}\\ \vdots \\ \vec{a}_m^T b\end{bmatrix} \qquad A\vec{x} = \vec{0}$$

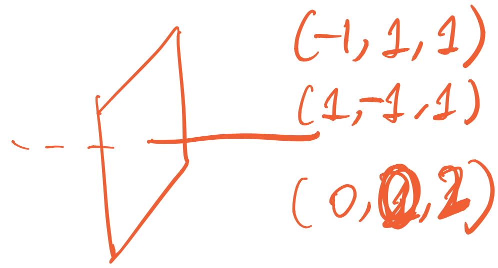
$\vec{a}\cdot\vec{b} = \vec{a}^T\vec{b}$ (matrix multiplication)

Def. The orthogonal complement of a nonempty subset S of $\mathbb{R}^n$, denoted by $S^\perp$ (read "S perp"), is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in S.

That is, $S^\perp = \{\vec{v} \in \mathbb{R}^n : \vec{v}\cdot\vec{u} = 0 \text{ for every } \vec{u} \text{ in } S\}$

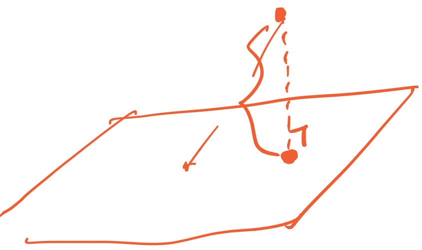
E.g. $(\mathbb{R}^n)^\perp = \{\vec{0}\}$, $\quad \{0\}^\perp = \mathbb{R}^n$,

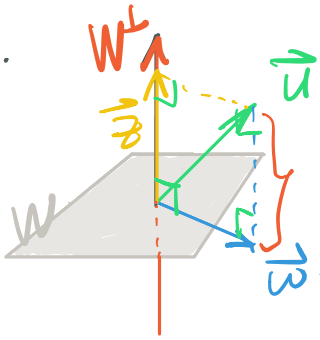




Note: the orthogonal complement of any nonempty subset of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$

Theorem (Orthogonal Decomposition Theorem) Let W be a subspace of $\mathbb{R}^n$. Then, for any vector $\vec{u}$ in $\mathbb{R}^n$, there exist unique vectors $\vec{w}$ in W and $\vec{z}$ in $W^{\perp}$ such that $\vec{u} = \vec{w} + \vec{z}$.





denoted $U_W(\vec{u})$

$\vec{w}$ = projection of $\vec{u}$ on W

= closest vector in W to $\vec{u}$

Distance between $\vec{u}$ and $\vec{w}$ = $\|\vec{u}-\vec{w}\| = \|\vec{z}\|$

More Related to Workshop 16 Problem 2b:

$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, then $A\vec{x} = \vec{b}$ is inconsistent.

Want to find $\vec{x}$ that minimizes $\|A\vec{x} - \vec{b}\|$.

1st Approach:

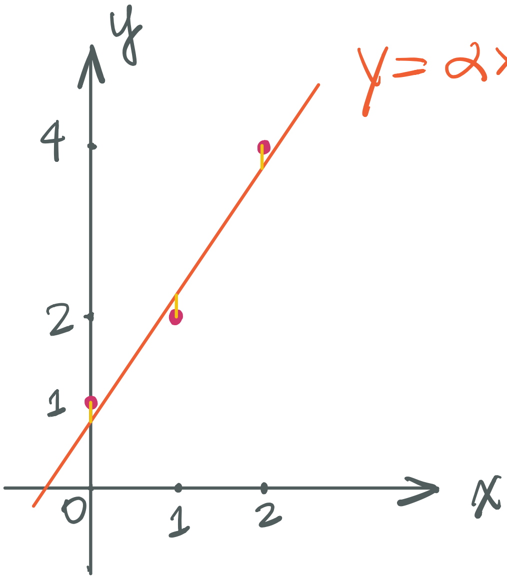
Let W = column span of A. $A\vec{x} = \vec{c}$ has a solution exactly when $\vec{c}$ is in W.

Let $\vec{w} = U_W(\vec{b})$, then $A\vec{x} = \vec{w}$ will have a solution $\vec{x}_0$, $\|A\vec{x}_0 - \vec{b}\| = \|\vec{w} - \vec{b}\|$ (this is as small as $\|A\vec{x} - \vec{b}\|$ can get!)

$\vec{0} = A^T\vec{z} = A^T(\vec{b} - \vec{w}) = A^T\vec{b} - A^T\vec{w} = A^T\vec{b} - A^TA\vec{x}_0$

So solve $A^TA\vec{x} = A^T\vec{b}$!

2nd Approach:
Use multivariable calculus.



$$\begin{cases} \alpha \cdot 0 + \beta = 1 \\ \alpha \cdot 1 + \beta = 2 \\ \alpha \cdot 2 + \beta = 4 \end{cases}$$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}}_{\vec{b}}$$

Def. A matrix A with real entries is called symmetric if $A^T = A$.

Theorem. Symmetric matrices are diagonalizable over $\mathbb{R}$.
* In particular, all eigenvalues of symmetric matrices are ~~real~~ real.

Theorem. If $\vec{u}$ and $\vec{v}$ are eigenvectors of a symmetric matrix A with distinct eigenvalues λ, μ, then $\vec{u} \cdot \vec{v} = 0$

Proof:

$A\vec{u} = \lambda \vec{u}$, $A\vec{v} = \mu \vec{v}$

$(A\vec{u}) \cdot \vec{v} = \lambda \vec{u} \cdot \vec{v}$, $\vec{u} \cdot A\vec{v} = \mu \vec{u} \cdot \vec{v}$

$\quad \| \qquad\qquad\qquad \|$

$(A\vec{u})^T \vec{v} = \vec{u}^T A^T \vec{v} = \vec{u}^T A \vec{v}$

$\Rightarrow \lambda \vec{u}\cdot\vec{v} = \mu \vec{u}\cdot\vec{v}$, i.e., $(\lambda-\mu)\vec{u}\cdot\vec{v}=0$ $\xrightarrow{\lambda-\mu\neq 0}$ $\vec{u}\cdot\vec{v}=0$

llary. Symmetric matrices are orthogonally diagonalizable.

Instructions

1. The statements in Italics are for introducing results and notations that may be used again in this course. You are only required to read and think about them.

2. To receive full credit you must explain how you got your answer.

3. While I encourage collaboration, you must write solutions IN YOUR OWN WORDS. DO NOT SHARE COMPLETE SOLUTIONS before they are due. YOU WILL RECEIVE NO CREDIT if you are found to have copied from whatever source or let others copy your solutions.

4. Workshops must be handwritten (electronic handwriting is allowed) for authentication purposes and submitted on Canvas. Please do NOT include any personal information such as your name and netID in your file. Late homework will NOT be accepted. It is your responsibility to MAKE SURE THAT YOUR SUBMISSIONS ARE SUCCESSFUL AND YOUR FILES ARE LEGIBLE AND COMPLETE. It is also your responsibility that whoever reads your work will understand and enjoy it. Up to 1 point out of 10 may be taken off if your solutions are hard to read or poorly presented.

Workshop 16

1. Let $\mathfrak{X} = \{\mathbf{u_1}, \mathbf{u_2}\}$, $\mathfrak{Y} = \{\mathbf{v_1}, \mathbf{v_2}\}$ be orthonormal bases for a 2−dimensional subspace V of $\mathbb{R}^n$.

a. Let $Q = [id_V]_{\mathfrak{X}\mathfrak{Y}}$. Express each entry of Q as a dot product.

b. Show that the columns of Q form an orthonormal set of vectors, and so are the rows.

orthogonal
length 1

☆ c. Show that b is equivalent to $Q^TQ = QQ^T = I_2$, i.e., $Q^{-1} = Q^T$.

d. Show that $(Q\mathbf{u}) \cdot (Q\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$ for any column vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$. (*We say Q preserves dot products.*)

use part c and

$\vec{a}\cdot\vec{b} = \underline{\vec{a}^T\vec{b}}$ (matrix multiplication)

2. Let W =Span$\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

a. Find an orthonormal basis $\mathfrak{X}$ for W and an orthonormal basis $\mathfrak{Y}$ for $W^\perp$. Verify that $\mathfrak{X} \cup \mathfrak{Y}$ is an orthonormal basis for $\mathbb{R}^3$.

b. Let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$. Find $\mathbf{w}$ in W and $\mathbf{z}$ in $W^\perp$ such that $\mathbf{b} = \mathbf{w} + \mathbf{z}$. (Hint: Problem 1 in Workshop 15 may be helpful.)

$$Q = \begin{bmatrix} \vec{v}_1\cdot\vec{u}_1 & \vec{v}_2\cdot\vec{u}_1 \\ \vec{v}_1\cdot\vec{u}_2 & \vec{v}_2\cdot\vec{u}_2 \end{bmatrix}$$

$$\vec{v}_1 = (\vec{v}_1\cdot\vec{u}_1)\vec{u}_1 + (\vec{v}_1\cdot\vec{u}_2)\vec{u}_2$$

$$\vec{v}_1\cdot\vec{v}_1 = (\vec{v}_1\cdot\vec{u}_1)^2 + (\vec{v}_1\cdot\vec{u}_2)^2$$

$$\vec{v}_1\cdot\vec{v}_1 = 1$$

$$(\vec{v}_1\cdot\vec{u}_1)^2 + (\vec{v}_1\cdot\vec{u}_2)^2 = 1$$

$$\vec{v}_1\cdot\vec{v}_1 = 1$$

A suppose columns of A are orthonormal.

rows

$$A^TA = I \qquad AA^T = I$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(1st column) $\cdot$ (1st column) $= 1$