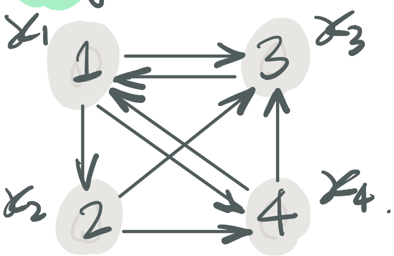
named after Google founder Larry Page

E.g. Google's PageRank Algorithm



$$\lim_{n\to\infty}(A^n \vec{x}) = \vec{y}$$

$$A\vec{y} = \vec{y}$$

$$A \cdot A^\infty \cdot \vec{x} = A^\infty \cdot \vec{x}$$

$$x_1 = x_3 + x_4 \cdot \tfrac{1}{2}$$
$$x_2 = x_1 \tfrac{1}{3}$$
$$x_3 = x_1 \tfrac{1}{3} + x_2 \cdot \tfrac{1}{2} + x_4 \cdot \tfrac{1}{2}$$
$$x_4 = x_1 \tfrac{1}{3} + x_2 \cdot \tfrac{1}{2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

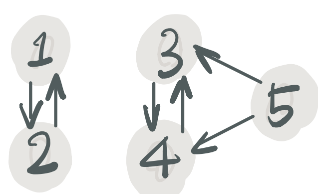
$$\underbrace{\begin{pmatrix} 25\% \\ 25\% \\ 25\% \\ 25\% \end{pmatrix}}_{\vec{x}} \longrightarrow \begin{bmatrix} 1 \cdot 25\% + \frac{1}{2} 25\% \\ \\ \\ \\ \end{bmatrix}$$

1 is an eigenvalue of A with eigenspace $E_1 = \text{span}\left\{\begin{bmatrix} 12 \\ 4 \\ 9 \\ 6 \end{bmatrix}\right\}$

Issues:

1. $\dim(E_1)$ can be greater than 1.

E.g.



$\dim(E_1) = 2$

2. A page with no outgoing link will create a column of 0's

$\longrightarrow$ 1 may not be an eigen values

Instructions

1. The statements in Italics are for introducing results and notations that may be used again in this course. You are only required to read and think about them.

2. To receive full credit you must explain how you got your answer.

3. While I encourage collaboration, you must write solutions IN YOUR OWN WORDS. DO NOT SHARE COMPLETE SOLUTIONS before they are due. YOU WILL RECEIVE NO CREDIT if you are found to have copied from whatever source or let others copy your solutions.

4. Workshops must be handwritten (electronic handwriting is allowed) for authentication purposes and submitted on Canvas. Please do NOT include any personal information such as your name and netID in your file. Late homework will NOT be accepted. It is your responsibility to MAKE SURE THAT YOUR SUBMISSIONS ARE SUCCESSFUL AND YOUR FILES ARE LEGIBLE AND COMPLETE. It is also your responsibility that whoever reads your work will understand and enjoy it. Up to 1 point out of 10 may be taken off if your solutions are hard to read or poorly presented.

Workshop 14

1. Let A be a square matrix such that the entries in each row sum to 1. Show that 1 is an eigenvalue of A.

2. Let A be a square matrix. Show that A and A^T have the same eigenvalues.

3. In the last example discussed during the lecture, suppose the people who own page 3 are infuriated by the fact that it's ranked lower than page 1. In an attempt to boost page 3's ranking, they create a page 5 that links to page 3, and page 3 also links to page 5. Does this boost page 3's ranking above that of page 1? (Hint: You can answer this question with little computation; certainly without getting an eigenvector for the new matrix.)

4. Prove: an orthogonal set of nonzero vectors is linearly independent.

5. Let $\{\mathbf{v_1}, ..., \mathbf{v_k}\}$ be an orthonormal basis for a subspace V of $\mathbb{R}^n$, show that $\mathbf{u} = (\mathbf{u} \cdot \mathbf{v}_1)\mathbf{v}_1 + ... + (\mathbf{u} \cdot \mathbf{v}_k)\mathbf{v}_k$ for every vector $\mathbf{u}$ in V. Explain how this equality makes intuitive sense.

6. Use the Gram-Schmidt process to find an orthogonal basis for

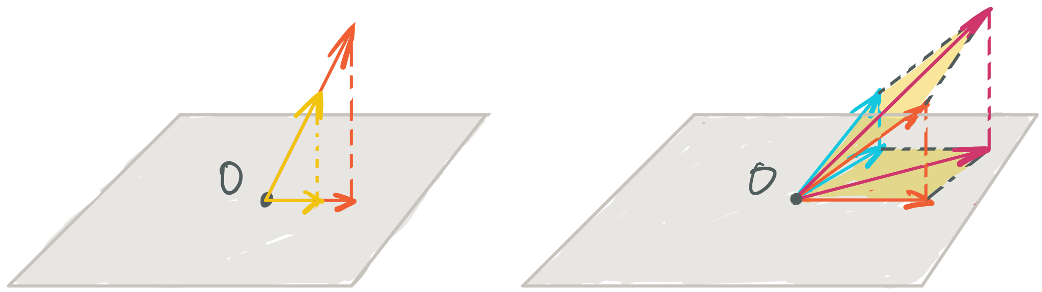
$$\text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right\}.$$

$$\begin{bmatrix} -1 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \vec{x} = \vec{0} \longrightarrow \begin{bmatrix} -1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & -1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & -1 & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -1 \end{bmatrix} \vec{x} = \vec{0}$$

The video makes use of the fact that projecting $\mathbb{R}^n$ onto a line through origin (1 dimensional subspace) is a linear transformation.

In general, projecting $\mathbb{R}^n$ onto a subspace is always a linear transformation.

E.g. Project $\mathbb{R}^3$ onto a plane (2-dimensional subspace)



The video used

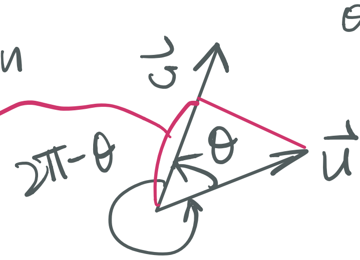
get a real number

vector $\vec{v}$ in $\mathbb{R}^n$ $\longleftrightarrow$ linear transformation $\vec{v}^*: \mathbb{R}^n \to \mathbb{R}$

project onto span$\{\vec{v}\}$ then scale by $\|\vec{v}\|$

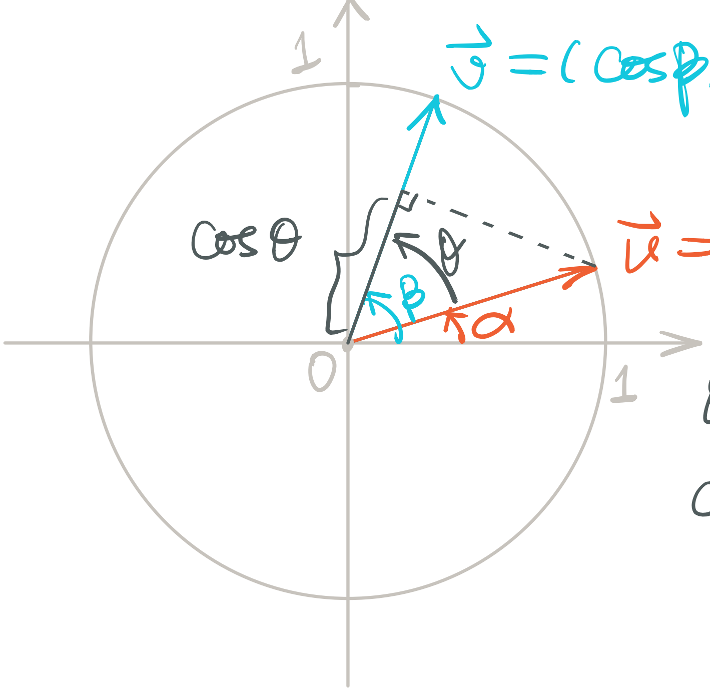
norm/Length of $\vec{v}$

to show $\vec{v}\cdot\vec{u} = \|\vec{v}\|\cdot\underbrace{\|\vec{u}\|\cos\theta}_{\text{projection}}$ ← angle from $\vec{u}$ to $\vec{v}$ or $\vec{v}$ to $\vec{u}$



$\|\vec{u}\|\cos\theta$

Another way to see it:



$$\theta = \beta - \alpha$$

$$\begin{aligned}\cos\theta &= \cos(\beta - \alpha) \\ &= \cos\beta\cos\alpha + \sin\beta\sin\alpha \\ &= \vec{v}\cdot\vec{u}\end{aligned}$$

In particular, $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{\vec{u}\cdot\vec{v}}{\sqrt{\vec{u}\cdot\vec{u}}\sqrt{\vec{v}\cdot\vec{v}}}$$

In $\mathbb{R}^n$, can use dot product to measure length of a vector and angle between vectors

$$\langle f, g \rangle = \int_a^b fg\,dx$$

Dot product on $\mathbb{R}^n$ generalizes to an inner product on an abstract vector space. Inner product produces the notion of length and angle.

For the sake of time, we'll not talk about inner product in general, at least for now.

Theorem (Cauchy-Schwatz Inequality) $|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\cdot\|\vec{v}\|$

Proof: $|\cos\theta| \le 1$

★ Being able to measure length and angle leads to the notions of unit vectors (i.e., vectors of length = 1) and orthogonality (i.e., $\vec{u} \cdot \vec{v} = 0$)

$\vec{0}$ is orthogonal to every vector;

for nonzero vectors $\vec{u}, \vec{v}$, this is equivalent to $\cos\theta = 0$, i.e., $\theta = \frac{\pi}{2} \pm 2n\pi$

Def. A set S of vectors is called orthogonal if the vectors in S are pairwise orthogonal, i.e., for any distinct vectors $\vec{u}$ and $\vec{v}$ in S, $\vec{u} \cdot \vec{v} = 0$.

If additionally, every vector in S has norm 1, then S is called orthonormal.

$\|\vec{v}\|$