Instructions

1. The statements in Italics are for introducing results and notations that may be used again in this course. You are only required to read and think about them.

2. To receive full credit you must explain how you got your answer.

3. While I encourage collaboration, you must write solutions IN YOUR OWN WORDS. DO NOT SHARE COMPLETE SOLUTIONS before they are due. YOU WILL RECEIVE NO CREDIT if you are found to have copied from whatever source or let others copy your solutions.

4. Workshops must be handwritten (electronic handwriting is allowed) for authentication purposes and submitted on Canvas. Please do NOT include any personal information such as your name and netID in your file. Late homework will NOT be accepted. It is your responsibility to MAKE SURE THAT YOUR SUBMISSIONS ARE SUCCESSFUL AND YOUR FILES ARE LEGIBLE AND COMPLETE. It is also your responsibility that whoever reads your work will understand and enjoy it. Up to 1 point out of 10 may be taken off if your solutions are hard to read or poorly presented.

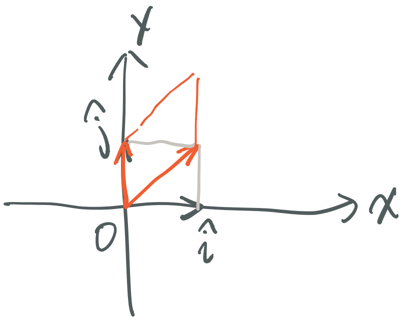
Workshop 11

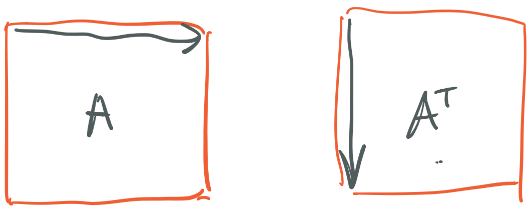
1. a. Compute the determinants of elementary matrices.
b. How does performing an elementary row or column operation on a matrix affect its determinant?

$A \longrightarrow EA$ $\det(EA) = \det(E)\det(A)$

2. How does transposing a matrix affect its determinant?

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ add row 1 to row 2





$\lambda^2(\lambda-1)^3(\lambda-2)$

eigenvalue. 0 1 2
multiplicity 2 3 1

Theorem. $\dim(E_\lambda) \le$ multiplicity of λ $\left(\begin{array}{c}\text{as a root of} \\ \det(A-\lambda I)\end{array}\right)$

Theorem. A set of eigenvectors of the same linear transformation T with distinct eigenvalues is linearly independent.

"Proof":

For simplity, we demonstrate the linear indepence of three eigenvectors u, v, w, with distinct eigenvalues α, β, γ, respectively.

Suppose $a\cdot u + b\cdot v + c\cdot w = 0$.

then $T(a\cdot u + b\cdot v + c\cdot w) = T(0) = 0$

$$\begin{aligned} T(a\cdot u + b\cdot v + c\cdot w) &= a\cdot T(u) + b\cdot T(v) + c\cdot T(w) \\ &= a\cdot\alpha\cdot u + b\cdot\beta\cdot v + c\cdot\gamma\cdot w \end{aligned}$$

$$(\cancel{a\cdot\alpha\cdot u} + b\cdot\beta\cdot v + c\cdot\gamma\cdot w) - \alpha(\cancel{a\cdot u} + b\cdot v + c\cdot w) = 0.$$

$$0. \qquad\qquad 0.$$

$$b(\beta-\alpha)\cdot v + c(\gamma-\alpha)\cdot w = 0.$$

$$T(b(\beta-\alpha)\cdot v + c(\gamma-\alpha)\cdot w) = T(0) = 0$$

$$\begin{aligned} T(b(\beta-\alpha)\cdot v + c(\gamma-\alpha)\cdot w) &= b(\beta-\alpha)\cdot T(v) + c(\gamma-\alpha)T(w) \\ &= b(\beta-\alpha)\beta v + c(\gamma-\alpha)\gamma\cdot w \end{aligned}$$

$$(\cancel{b(\beta-\alpha)\beta v} + c(\gamma-\alpha)\gamma\cdot w) - \beta(\cancel{b(\beta-\alpha)\cdot v} + c(\gamma-\alpha)\cdot w) = 0$$

$$0 \qquad\qquad 0$$

$$c(\gamma-\alpha)(\gamma-\beta)w = 0 \implies c = 0$$

α, β, γ distinct $\quad \cancel{=} 0$ (number) $\quad \neq 0$ (vector)

$(\lambda^2+1)(\lambda-2)$

Can similarly show $a, b = 0$

$\lambda^2(\lambda-1)^3(\lambda-2)$ deg $6 = \dim(V)$

$2 \to E_0 \quad 3 \to E_1 \quad E_2 \leftarrow 1$

Corollary: When ① the characteristic polynomial has $\dim(V)$ $= n$ roots (in $\mathbb{R}$ if we are working over $\mathbb{R}$) (counting multiplicity), and

② $\dim(E_\lambda) =$ multiplicity of λ for each eigenvalue λ,

we can get a basis $\mathfrak{X}$ for V consisting of eigenvectors, (eigenbasis)

and $[T]_{\mathfrak{X}\mathfrak{X}}$ will be a diagonal matrix.

Def. In this case, we say T is diagonalizable.

An $n \times n$ matrix is called diagonalizable if it is similar to a diagonal matrix. (That is, an $n \times n$ matrix A is diagonalizable if there exists an $n \times n$ invertible matrix P and an $n \times n$ diagonal matrix D such that $A = P^{-1}DP$.)

✱ A is diagonalizable if and only if $A = [T]_{\mathfrak{X}\mathfrak{X}}$ for some diagonalizable linear transformation T under some basis $\mathfrak{X}$.

Applications of Eigenstuff

E.g. System of differential equations:

find differentiable functions $x_1(t), x_2(t)$ such that

$$\begin{cases} x_1' = x_1 + x_2 \\ x_2' = 4x_1 + x_2 \end{cases} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

QDQ^{-1}

$P^{-1}DP$

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

multiply $\begin{bmatrix}1 & 1\\ 2 & -2\end{bmatrix}^{-1}$ on the left $\Rightarrow$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x_1'\\ x_2'\end{pmatrix} = \begin{pmatrix}ax_1'+bx_2'\\ cx_1'+dx_2'\end{pmatrix}$$

$$\begin{bmatrix}1 & 1\\ 2 & -2\end{bmatrix}^{-1}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}' = \begin{bmatrix}3 & 0\\ 0 & -1\end{bmatrix}\cdot\Bigg(\underbrace{\begin{bmatrix}1 & 1\\ 2 & -2\end{bmatrix}^{-1}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{\begin{bmatrix}y_1\\ y_2\end{bmatrix}}\Bigg)$$

$$\| $$

$$\Bigg(\underbrace{\begin{bmatrix}1 & 1\\ 2 & -2\end{bmatrix}^{-1}\begin{bmatrix}x_1\\ x_2\end{bmatrix}}\Bigg)'$$

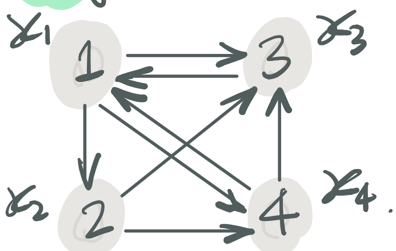
$$\left(\begin{bmatrix}y_1\\ y_2\end{bmatrix}\right)' = \begin{bmatrix}3 & 0\\ 0 & -1\end{bmatrix}\cdot\begin{bmatrix}y_1\\ y_2\end{bmatrix}$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 1\\ 2 & -2\end{bmatrix}^{-1}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{\begin{bmatrix}ax_1+bx_2\\ cx_1+dx_2\end{bmatrix}}$$

$$\begin{cases} y_1' = 3\cdot y_1 & e^{3t}\\ y_2' = -1\cdot y_2 & e^{-t}\end{cases}$$

named after Google founder Larry Page

E.g. Google's PageRank Algorithm



$$x_1 = x_3 + x_4\cdot\tfrac{1}{2}$$
$$x_2 = x_1\tfrac{1}{3}$$
$$x_3 = x_1\tfrac{1}{3} + x_2\cdot\tfrac{1}{2} + x_4\cdot\tfrac{1}{2}$$
$$x_4 = x_1\tfrac{1}{3} + x_2\cdot\tfrac{1}{2}$$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 0 & 1 & \frac{1}{2}\\ \frac{1}{3} & 0 & 0 & 0\\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2}\\ \frac{1}{3} & \frac{1}{2} & 0 & 0\end{bmatrix}}_{A.}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

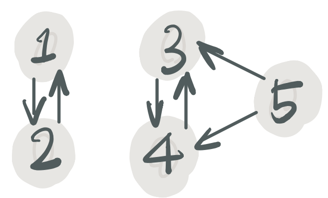
① A A^T $A-\lambda I$ $A^T-\lambda I$ ✓

②

1 is an eigenvalue of A with eigenspace $E_1 = \text{span}\left\{\begin{bmatrix}12\\ 4\\ 9\\ 6\end{bmatrix}\right\}$

Issues:

1. $\dim(E_1)$ can be greater than 1.

E.g.  $\dim(E_1) = 2$

2. A page with no outgoing link will create a column of 0's

$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} = P^{-1} D P$?

$\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$

$\mathcal{Y}$: standard basis.

$A = [T]_{\mathcal{Y}\mathcal{Y}}$

want to find an eigenbasis $\mathcal{X}$. $[T]_{\mathcal{X}\mathcal{X}} = D$.

Compute eigenvalues: $\boxed{\lambda = 3, \lambda = -1}$ $\quad \mathcal{X} = \{\overset{3}{(1,2)}, \overset{-1}{(1,-2)}\}$

$A - \lambda I \qquad (A - 3I)\vec{v} = 0 \qquad (A + I)\cdot\vec{v} = 0.$

$\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$

eigenbasis

$1\cdot(1,0) + 2(0,1) = (1,2) \qquad 1(1,0) - 2(0,1) = (1,-2)$

$[T]_{\mathcal{Y}\mathcal{Y}} = [id_{\mathbb{R}^2}]_{\mathcal{Y}\mathcal{X}} \, [T]_{\mathcal{X}\mathcal{X}} \, [id_{\mathbb{R}^2}]_{\mathcal{X}\mathcal{Y}}$

$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1} \qquad [id_{\mathbb{R}^2}]_{\mathcal{Y}\mathcal{X}} =$

$= \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}^{-1}$

$\mathcal{Y} = \{(1,0),(0,1)\}$ $\qquad$ $\mathcal{X} = \{(1,2),(1,-2)\}$

$$[id_{\mathbb{R}^2}]_{\mathcal{Y}\mathcal{X}} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

$$[id_{\mathbb{R}^2}]_{\mathcal{X}\mathcal{Y}}$$

$\mathcal{Y}$

$$id((1,2)) = (1,2) = 1\cdot(1,0) + 2(0,1)$$

$$\mathcal{X} = \{(1,-2),(1,2)\}$$

$$\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$$

λ_1		
0		
0		