Def. Let V be a vector space over $\mathbb{R}/\mathbb{C}$ and $T: V \to V$ a linear transformation. A nonzero vector v in V is called an eigenvector of T with eigenvalue λ if $T(v) = \lambda \cdot v$ for some λ in $\mathbb{R}/\mathbb{C}$.

E.g. Zero map, id.

number 0.

$$\underset{\text{zero map}}{0}(v) = \underset{\text{zero vector}}{0} = 0 \cdot v$$

$id_V: V \to V$

$v \mapsto v$.

☆ When a basis $\mathcal{X}$ for V is chosen, V is identified with $\mathbb{R}^n/\mathbb{C}^n$ (n: dim(V)), and $[T]_{\mathcal{X}\mathcal{X}}$ is an $n \times n$ matrix.

Def. Let A be an $n \times n$ matrix with entries in $\mathbb{R}/\mathbb{C}$, a nonzero vector $\vec{v}$ in $\mathbb{R}^n/\mathbb{C}^n$ is called an eigenvector of A with eigenvalue λ if $A\vec{v} = \lambda\vec{v}$ for some λ in $\mathbb{R}/\mathbb{C}$.

write it as a column vector (i.e., $n \times 1$ matrix)

☆ A is invertible if and only 0 is not an eigenvalue of A

" A is not invertible $\Leftrightarrow$ 0 is an eigenvalue of A "

0 is an eigenvalue of A: exists a nonzero vector $\vec{v}$ such that $A \cdot \vec{v} = 0 \cdot \vec{v} = \vec{0}$

$$A\vec{x} = \vec{0}$$

Let V be a vector space over $\mathbb{R}/\mathbb{C}$ with basis $\mathcal{Y}$ and $T: V \to V$ a linear transformation. We can express each eigenvector of T as a linear combination of the vectors in $\mathcal{Y}$.

E.g.

Workshop 7

1. a. Let $\mathfrak{X}$ be the **standard basis** $\{(1,0,0),(0,1,0),(0,0,1)\}$ of $\mathbb{R}^3$. In Workshop 3 we saw that $\mathfrak{Y} = \{(1,1,0),(0,0,1),(1,0,1)\}$ is another basis. Write down $[id_{\mathbb{R}^3}]_{\mathfrak{Y}\mathfrak{X}}, [id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{Y}}$ (*these are called* ***change of basis matrices***) and compute $[id_{\mathbb{R}^3}]_{\mathfrak{Y}\mathfrak{X}}[id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{Y}}, [id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{Y}}[id_{\mathbb{R}^3}]_{\mathfrak{Y}\mathfrak{X}}$. *In general, change of basis matrices are invertible, and every invertible matrix is a change of basis matrix.*

b. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation with $[T]_{\mathfrak{X}\mathfrak{X}} = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$.

$(1-\lambda)(2-\lambda)(3-\lambda)$

Compute $[T]_{\mathfrak{Y}\mathfrak{X}}, [T]_{\mathfrak{X}\mathfrak{Y}}, [T]_{\mathfrak{Y}\mathfrak{Y}}$.

$$[T]_{yy} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad -(\lambda-1)(\lambda-2)(\lambda-3)$$

Can express eigenvectors $(1,0,0), (0,1,0), (0,0,1)$ each as a linear combination of the vectors in y:

$(1,0,0) = 0\cdot(1,1,0) - 1\cdot(0,0,1) + 1\cdot(1,0,1)$ $\quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$(0,1,0) = 1\cdot(1,1,0) + 1\cdot(0,0,1) - 1\cdot(1,0,1)$ $\quad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$(0,0,1) = 0\cdot(1,1,0) + 1\cdot(0,0,1) + 0\cdot(1,0,1)$ $\quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = 1\cdot\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = 2\cdot\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\vdots$

Question: If only given $[T]_{yy}$, how do we express eigenvectors as linear combinations of the vectors in y?

Let $A = [T]_{\mathcal{Y}\mathcal{Y}}$, then $A\vec{v} = \lambda \vec{v}$

$$\lambda \vec{v} = \lambda \cdot I \cdot \vec{v}$$

$(A - \lambda \cdot I)\vec{v} = \vec{0}$ ① Look for solutions $\vec{v} \neq \vec{0}$

Such solutions exist exactly when $\det(A - \lambda \cdot I) = 0$! ②

e.g. $\lambda^3 - 2\lambda + 1$

Strategy: Find eigenvalues λ from ②, then plug them into ① to solve for $\vec{v}$.

✱ For each eigenvalue λ, get a vector space of solutions to ① (called the eigenspace of λ, denoted E_λ); find a basis for each eigenspace.

✱ When V is a vector space of dimension n, A is $n \times n$, and $\det(A - \lambda \cdot I)$ is a polynomial (in λ) of degree n. It's called the characteristic polynomial of A.

$$P^{-1}\lambda I P = \lambda P^{-1} I P = \lambda P^{-1}P = \lambda I$$

✱ Let $B = [T]_{\mathcal{X}\mathcal{X}}$ for another basis $\mathcal{X}$ for V, then $B = P^{-1}AP$ for some invertible matrix P.

$$B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - P^{-1}\lambda I P = P^{-1}(A - \lambda I)P$$

$$\Rightarrow \det(B - \lambda I) = \det(P^{-1}(A - \lambda I)P) = \det(P^{-1})\det(A - \lambda I)\det(P)$$

$$= \det(A - \lambda I)(\det(P^{-1})\det(P))$$

$$= \det(A - \lambda I)\det(P^{-1}P)$$

$$\det(A - \lambda I) = \det(A - \lambda I)\,\boxed{\det(I)}, \quad \det(I) = 1$$

i.e., similar matrices have the same characteristic polynomial

So this polynomial is also called the characteristic polynomial of T.

$\lambda^3 = 0 \rightarrow (\lambda - 0)^3$ $\quad 0, 0, 0$

* By the fundamental theorem of algebra, it will always have n complex roots (counted with multiplicity), some of which may not be real.