Theorem. An $n \times n$ matrix A (with real entries) is invertible ①

= Its reduced row/column echelon form is I_n ②

= It is a product of elementary matrices ③

= Its determinant is nonzero ④

= It's a change of basis matrix $[id]_{\mathcal{X}\mathcal{Y}}$ $[id]_{\mathcal{Y}\mathcal{X}}$ ⑤

= Its columns form a basis for $\mathbb{R}^n$ ⑥

= It's full-rank (i.e., has biggest rank possible). ⑦

= Its rows form a basis for $\mathbb{R}^n$ ⑧

= The only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$ ⑨

= 0 is not an eigenvalue for A

Proof:

$$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \qquad \begin{bmatrix} I & 0 \\ 0 & \end{bmatrix}$$

zero det = columns linearly dependent
$\qquad$ ||
not invertible.

$$A\vec{x} = \vec{0}$$

$$(A^{-1} \cdot A)\vec{x} = A^{-1} \cdot \vec{0} = \vec{0}.$$
$$I \cdot \vec{x}$$
$$\vec{x}.$$

$$A\vec{x} = x_1 \begin{bmatrix} 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \end{bmatrix} + x_4 \begin{bmatrix} 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0}.$$

$\{ 2x_1 + x_2 = 3$

1

assume $\det(A) = 0$

$$A\vec{x} = \vec{b} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$c_1(a, b) + c_2(c, d) = (0, 0)$

$$\left[\begin{array}{cc|c} a & b & b_1 \\ c & d & b_2 \end{array}\right]$$

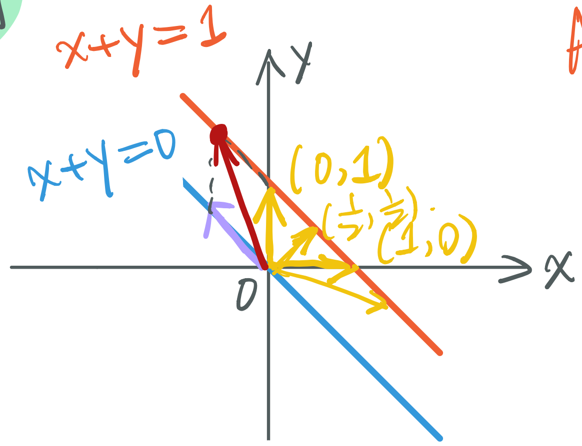
$c_1 b_1 + c_2 b_2 = 0$ infinite solutions no solution

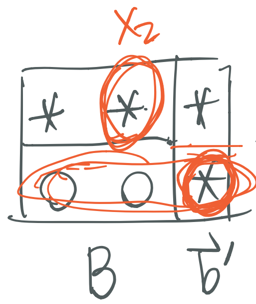
✱ Solutions to homogeneous vs. inhomogeneous equations:

E.g.

$A\vec{x}=\vec{b}\neq\vec{0}$

$A\vec{x}=\vec{0}$





$A\vec{x}=\vec{b}$

$B\vec{x}=\vec{b}'$

$A\vec{x}_0=\vec{b}$ ✓

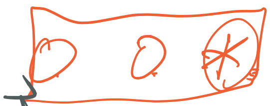
$A\vec{x}=\vec{0}$.

$A(\vec{x}_0+\vec{x})=\vec{b}$

$A\vec{x}=\vec{b}$

$A\vec{x}=\vec{c}$

$A\vec{x}=\vec{d}$



In general, solutions to an inhomogeneous system of equations are obtained by adding a special solution to the inhomoge. to the corresponding homogeneous system of equations.

✱ When are (differentiable) functions linearly independent?

E.g. Let f and g be differentiable functions

Suppose $a\cdot f+b\cdot g=0$ for a,b in $\mathbb{R}$,

then $a\cdot f'+b\cdot g'=0$, and therefore $a\cdot\begin{bmatrix}f\\f'\end{bmatrix}+b\cdot\begin{bmatrix}g\\g'\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

This means:

f and g are linearly dependent $=$

$\begin{bmatrix}f(x)\\f'(x)\end{bmatrix}$ and $\begin{bmatrix}g(x)\\g'(x)\end{bmatrix}$ are linearly dependent for all x.

$$= \det\left(\begin{bmatrix}f(x) & g(x)\\ f'(x) & g'(x)\end{bmatrix}\right)=0 \text{ for all } x$$

$\parallel$

$W[f,g](x)$ Wronskian.

In particular:

If $W[f,g](x)$ is not identically 0, then f and g are linearly independent.

★ Review: Matrices representing a linear transformation under different bases

$T: V \to V$, $\mathfrak{X}, \mathfrak{Y}$ bases for V

Relationship between $[T]_{\mathfrak{X}\mathfrak{X}}$ and $[T]_{\mathfrak{Y}\mathfrak{Y}}$?

$T = id_V \circ T \circ id_V$

$$[T]_{\mathfrak{X}\mathfrak{X}} = [id_V \circ T \circ id_V]_{\mathfrak{X}\mathfrak{X}}$$
$$= [id_V]_{\mathfrak{X}\mathfrak{Y}}[T]_{\mathfrak{Y}\mathfrak{Y}}\underbrace{[id_V]_{\mathfrak{Y}\mathfrak{X}}}_{P}$$

$$P^{-1}[T]_{\mathfrak{Y}\mathfrak{Y}}P = [T]_{\mathfrak{X}\mathfrak{X}}.$$
$$\neq$$
$$[T]_{\mathfrak{Y}\mathfrak{Y}}P^{-1}P$$
$$\|$$
$$C$$

$$[T]_{\mathfrak{Y}\mathfrak{Y}} = [id_V \circ T \circ id_V]_{\mathfrak{Y}\mathfrak{Y}}$$
$$= [id_V]_{\mathfrak{Y}\mathfrak{X}}[T]_{\mathfrak{X}\mathfrak{X}}[id_V]_{\mathfrak{X}\mathfrak{Y}}$$

Def. Two $n\times n$ square matrices A and B are called similar if there exists an $n\times n$ invertible matrix P such that $B = P^{-1}AP$

$\Leftrightarrow$ $A = [T]_{\mathfrak{X}\mathfrak{X}}$ and $B = [T]_{\mathfrak{Y}\mathfrak{Y}}$ for some linear transformation $T: V \to V$ and $\mathfrak{X}, \mathfrak{Y}$ bases for V

Eigenvectors and Eigenvalues

E.g. (Workshop 7 Problem 1b)

$T(v_1) = 1\cdot v_1 + 0\cdot v_2 + 0\cdot v_3$

$T(v_2) = 0\cdot v_1 + 2\cdot v_2 + 0\cdot v_3$

b. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation with $[T]_{\mathfrak{X}\mathfrak{X}} = \begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$.

$\{v_1, v_2, v_3\} = \mathfrak{X}$ consists of eigenvectors

Compute $[T]_{\mathfrak{Y}\mathfrak{X}}, [T]_{\mathfrak{X}\mathfrak{Y}}, [T]_{\mathfrak{Y}\mathfrak{Y}}$.

$$[T]_{\mathfrak{Y}\mathfrak{Y}} = \begin{bmatrix}2&0&0\\1&3&2\\-1&0&1\end{bmatrix} = P^{-1}[T]_{\mathfrak{X}\mathfrak{X}}P.$$

Def. Let V be a vector space over $\mathbb{R}/\mathbb{C}$ and $T: V \to V$ a linear transformation. A nonzero vector v in V is called an **eigenvector** of T with **eigenvalue** λ if $T(v) = \lambda \cdot v$ for some λ in $\mathbb{R}/\mathbb{C}$.

E.g. Zero map, id.

number 0.

$$0(v) = 0 = 0 \cdot v$$

zero map zero vector

$$id_V : V \to V$$
$$v \mapsto v.$$

	$A\vec{x} = \vec{0}$	$A\vec{x} = \vec{b} \neq 0$
$\det(A) \neq 0$	unique solution $\vec{x} = \vec{0}$	✓ unique solution $\vec{x} = A^{-1} \vec{b}$
$\det(A) = 0$	infinite solutions	✓ infinite solutions / no solution.

$A\vec{x_0} = \vec{b}$

$$A \xrightarrow{\text{RREF}} \begin{array}{ccc} x_1 & x_2 & x_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}$$