Comments on workshops:

☆ A vector space over $\mathbb{C}$ is a vector space over $\mathbb{R}$

$v + u \qquad a \cdot v$

$\uparrow$

in $\mathbb{C}$.

$\mathbb{R}^2$

☆ Subspace: contains zero vector, closed under $+, \cdot$ $\quad u+v$

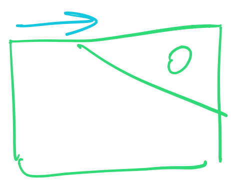
Linear transformation: preserves $+, \cdot \qquad T(u+v) = T(u) + T(v)$

☆ Workshop 10:

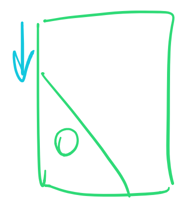
$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & -3 & y_1 \\ 2 & -4 & 2 & 0 & 8 & y_2 \\ 1 & -2 & 3 & -3 & 16 & y_3 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 1 & -3y_1 + 3y_2 - 2y_3 \\ 0 & 0 & 1 & 0 & 3 & 3y_1 - \frac{5}{2}y_2 + 2y_3 \\ 0 & 0 & 0 & 1 & -2 & 2y_1 - \frac{3}{2}y_2 + y_3 \end{array}\right]$$

For which values of y_1, y_2, y_3 does

$$\begin{cases} 1x_1 + 2x_2 + 1x_5 = -3y_1 + 3y_2 - 2y_3 \\ 1x_3 + 3x_5 = 3y_1 - \frac{5}{2}y_2 + 2y_3 \\ 1x_4 - 2x_5 = 2y_1 - \frac{3}{2}y_2 + y_3 \end{cases}$$



have a solution? ANS: For all values.

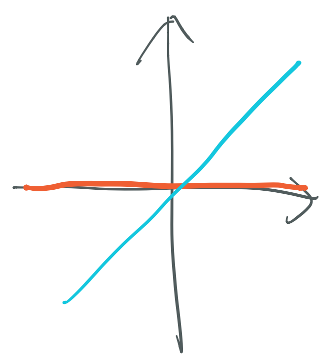


Row operation does NOT preserve column span!

E.g. $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ $\quad$ column span $= \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

RREF $\downarrow \qquad\qquad \nparallel$

$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $\quad$ column span $= \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$



However, pivot columns in original matrix form a basis for column span. This follows from the column correspondence property in the textbook.

$$\begin{tikzcd} W \arrow[r, "\det(T)"] & W \\ V \arrow[u] \arrow[r, "T"] & V \arrow[u] \end{tikzcd}$$

1 dim.

$\mathscr{X} = \{(1,0),(0,1)\}$ $\quad \mathscr{Y} = \{(\frac{1}{2},0),(0,\frac{1}{2})\}$

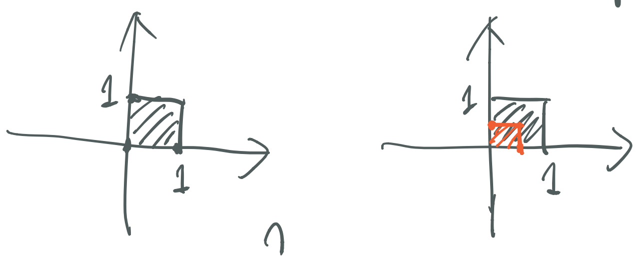
$T: V \longrightarrow V$

$\det([id]_{\mathscr{Y}\mathscr{X}}) = 4 = \dfrac{1}{\frac{1}{2}\times\frac{1}{2}}$

$\mathbb{R}^2 \qquad \hat{i} = (1,0), \quad \hat{j} = (0,1) \qquad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\mathbb{R}^3 \qquad \hat{i} = (1,0,0), \quad \hat{j} = (0,1,0), \quad \hat{k} = (0,0,1).$

Does the determinant depend on the choice of bases?



$\det([id]_{\mathscr{X}\mathscr{X}}) = 1$

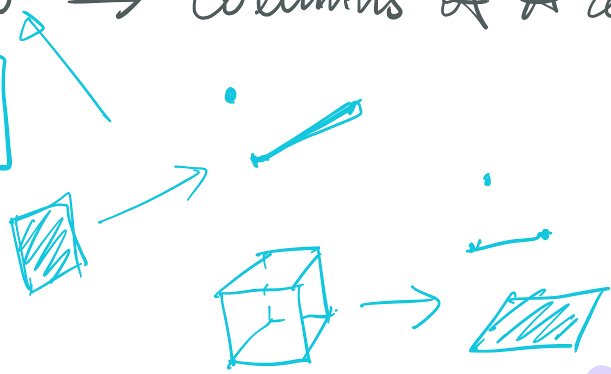
$\parallel$

$\det([id]_{\mathscr{Y}\mathscr{Y}}) = 1$

No as long as use the same basis for domain and target.

$\det(A) = 0 \overset{?}{\Longrightarrow}$ Columns of A are linearly dependent

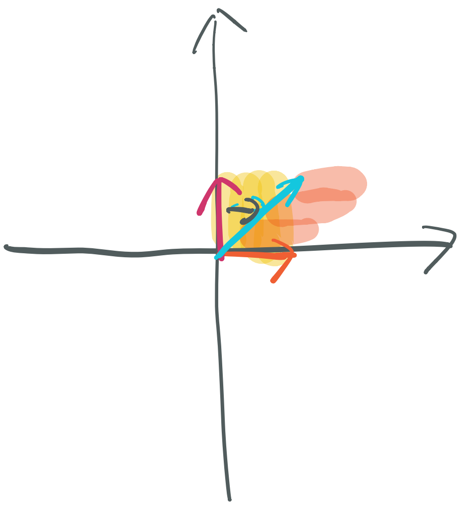
$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

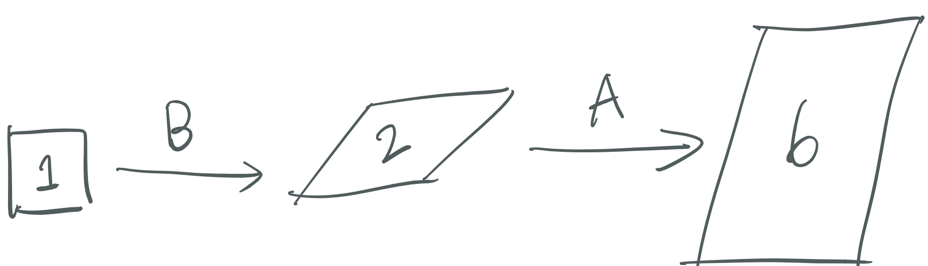


range = column span

$\det(AB) = \det(A)\det(B)$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$





$\det(B) = 2$

$\det(A) = 3$

$$\det([a]) = a \qquad \det([a_{22}]) \qquad \det([a_{21}])$$

$$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \underbrace{a_{11}a_{22}}_{(-1)^{1+1}} - \underbrace{a_{12}a_{21}}_{(-1)^{1+2}}$$

$$= -\underbrace{a_{21}a_{12}}_{(-1)^{2+1}} + \underbrace{a_{22}a_{11}}_{(-1)^{2+2}}$$

$$= (-1)^{1+1} a_{11}a_{22} - (-1)^{2+1} a_{21}a_{12}$$

$$= (-1)a_{12}a_{21} + a_{22}a_{11}$$

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = (-1)^{1+1} a_{11} \det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) + (-1)^{1+2} a_{12} \cdot \det\left(\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}\right) + (-1)^{1+3} a_{13} \cdot \det\left(\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{31} \end{bmatrix}\right)$$

In general, $\overset{n\times n}{\det(A)} = (-1)^{i+1} a_{i1} A_{i1} + \cdots + (-1)^{i+n} \overset{a_{in}}{A_{in}}$ ← expand along i-th row

$= (-1)^{1+j} a_{1j} A_{1j} + \cdots + (-1)^{n+j} \underset{a_{nj}}{A_{nj}}$ ← j-th column

E.g.

$$\det\left(\begin{bmatrix} -2 & 0 & 1 & 4 \\ -1 & 0 & 0 & 2 \\ 97 & 3 & 103 & -89 \\ 1 & 0 & -3 & 5 \end{bmatrix}\right) = (-1)^{3+1} \cdot 3 \cdot \det\left(\begin{bmatrix} -2 & 1 & 4 \\ -1 & 0 & 2 \\ 1 & -3 & 5 \end{bmatrix}\right)$$

(expand along 2nd column)

$$= 3\left[(-1)^{2+1}(-1)\det\left(\begin{bmatrix} 1 & 4 \\ -3 & 5 \end{bmatrix}\right) + (-1)^{2+3}\, 2 \det\left(\begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}\right)\right]$$

$$= \cdots$$

Theorem. $\det(AB) = \det(A)\det(B)$

$$\mathrm{Cof} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$\forall$ $A^{-1} = \frac{1}{\det(A)} \cdot [\mathrm{Cof}(A)]^T$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $\mathrm{Cof}(A)$ is the matrix with (i,j)-th entry $(-1)^{i+j} A_{ij}$

det of matrix obtained by removing ith row jth column