E.g. Let $T: \mathbb{R}^5 \to \mathbb{R}^3$ be given by the matrix

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix}$$

under standard bases. Compute $K(T)$, $R(T)$.

$\mathcal{X} = \{(1,0,0), (0,1,0), (0,0,1)\}$ standard basis for $\mathbb{R}^3$

$\mathcal{Y} = \{(1,0,0,0,0), \cdots, (0,0,0,0,1)\}$ standard basis for $\mathbb{R}^5$

$[T]_{\mathcal{X}\mathcal{Y}}$ $\underline{v = (x_1, x_2, x_3, x_4, x_5)}$ in $\mathbb{R}^5$,

$$\begin{bmatrix} \quad \\ \hline \\ \hline \\ \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \longrightarrow \boxed{\begin{matrix} T(v) = \\ y_1 \cdot (1,0,0) + \\ y_2 \cdot (0,1,0) + \\ y_3 (0,0,1). \end{matrix}} = (y_1, y_2, y_3) \text{ in } \mathbb{R}^3$$

$K(T) =$ the set of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ in $\mathbb{R}^5$ such that

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

i.e., $K(T)$ is given by the set of real solutions to

$$\begin{cases} 1x_1 + (-2)x_2 + 0x_3 + 2x_4 + (-3)x_5 = 0 \\ 2x_1 + (-4)x_2 + 2x_3 + 0x_4 + 8x_5 = 0 \\ 1x_1 + (-2)x_2 + 3x_3 + (-3)x_4 + 16x_5 = 0 \end{cases}$$

The 3 elementary row operations on the matrix

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix}$$

correspond to switching 2 equations, scaling one equation by a real number, and adding a real-multiple of one equation to another. Therefore, they do not change the set of solutions.

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix} \begin{matrix} \\ +(-2)\cdot \text{row } 1 \\ +(-1)\cdot \text{row } 1 \end{matrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 2 & -4 & 14 \\ 0 & 0 & 3 & -5 & 19 \end{bmatrix} \begin{matrix} \\ \times \frac{1}{2} \\ \\ \end{matrix}$$

this process is called Gaussian Elimination

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 & 7 \\ 0 & 0 & 3 & -5 & 19 \end{bmatrix} \begin{matrix} \\ \\ +(-3)\cdot \text{row } 2 \end{matrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{matrix} +(-2)\cdot \text{row } 3 \\ +2\cdot \text{row } 3 \\ \\ \end{matrix}$$

a) $(1, -2, 0, 0, 1)$

b) $(0, 0, 1, 0, 3)$

c) $(0, 0, 0, 1, -2) = (0, 0, 0, 0, 0)$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

← reduced row echelon form of the matrix

$\circ \quad x_2 \quad \circ \quad \circ \quad x_5$ are free variables → can take any real value

$$\begin{aligned} 1x_4 + (-2)x_5 = 0 &\Rightarrow x_4 = 2x_5 \\ 1x_3 + 3x_5 = 0 &\Rightarrow x_3 = -3x_5 \\ 1x_1 + (-2)x_2 + 1x_5 = 0 &\Rightarrow x_1 = 2x_2 - x_5 \end{aligned}$$

, x_2, x_5 any real value

→ general solution.

$$K(T) = \left\{ \begin{bmatrix} 2x_2 - x_5 \\ x_2 \\ -3x_5 \\ 2x_5 \\ x_5 \end{bmatrix}, \ x_2, x_5 \in \mathbb{R} \right\}$$

(subspace of $\mathbb{R}^5$)

dimension = number of vectors in a basis

basis : subset ① linearly independent
② span $K(T)$

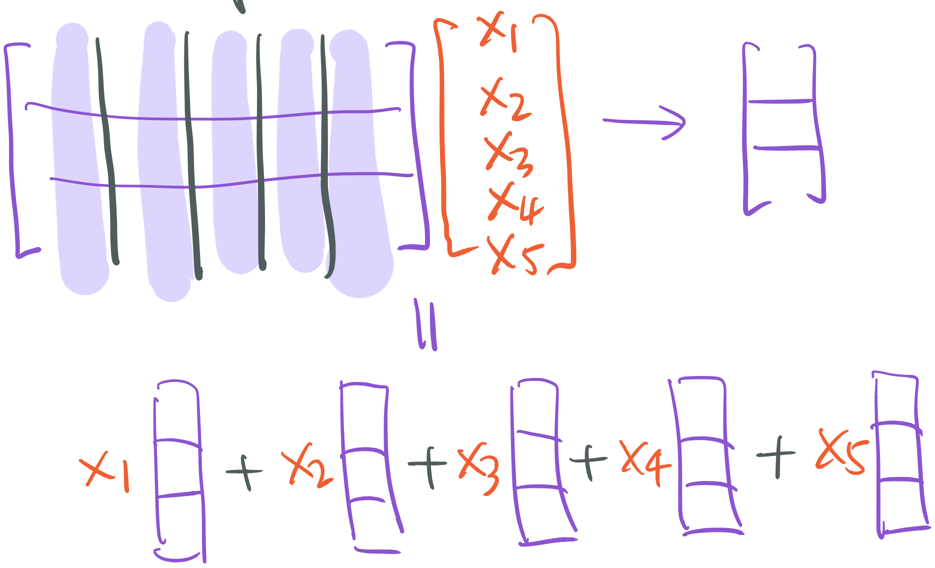
$$\begin{bmatrix} 2x_2 - x_5 \\ x_2 \\ -3x_5 \\ 2x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

span $K(T)$

$$a \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2a-b \\ a \\ -3b \\ 2b \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a=b=0 \Rightarrow \left\{ \begin{bmatrix} \\ \end{bmatrix}, \begin{bmatrix} \\ \end{bmatrix} \right\}$$ linearly indep.

☆ Note that $R(T)$ is exactly the span of the column vectors of the matrix.

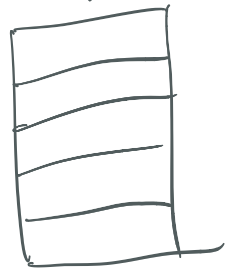


☆ Elementary row operations preserve row span
column column

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix}$$

A

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -4 & -2 \\ 0 & 2 & 3 \\ 2 & 0 & -3 \\ -3 & 8 & 16 \end{bmatrix} \xrightarrow{\text{Gaussian Elimination}} \text{RREF.}$$



A^T (A transpose).

$R(T) = \mathbb{R}^3 \quad \dim R(T) = 3 \quad \dim K(T) = 2$

$3 + 2 = 5$

Dimension Theorem: V, W vector spaces over $\mathbb{R}/\mathbb{C}$, $T: V \to W$ linear transformation, then $\dim V = \dim K(T) + \dim R(T)$

$\dim K(T)$ = nullity of T

$\dim R(T)$ = rank of T = dimension of column span of the matrix = dimension of row span of the matrix

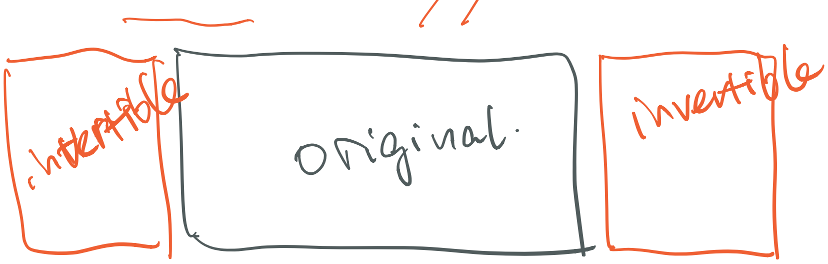
$\dim R(T) = \dim V - \dim K(T)$

rank of the matrix = dimension of column span of the matrix = dimension of row span of the matrix

$$\begin{bmatrix} 1 & * & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

dim(column span) = dim(row span)

= # 1's

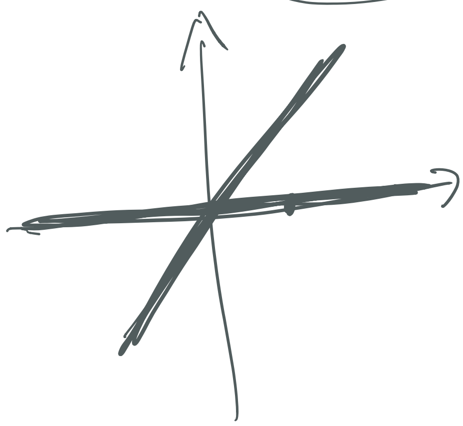


$$T \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$\mathbb{R}^2 \to \mathbb{R}^2$

$$\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

$$\underbrace{\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}}_{R(T)}$$



$$\begin{bmatrix} \boxed{1 \quad 1} \\ 0 \quad 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Definitions A matrix is said to be in **row echelon form** if it satisfies the following three conditions:

1. Each nonzero row lies above every zero row.
2. The leading entry of a nonzero row lies in a column to the right of the column containing the leading entry of any preceding row.
3. If a column contains the leading entry of some row, then all entries of that column below the leading entry are 0.[5]

If a matrix also satisfies the following two additional conditions, we say that it is in **reduced row echelon form**.[6]

4. If a column contains the leading entry of some row, then all the other entries of that column are 0.
5. The leading entry of each nonzero row is 1.

It can be shown that the reduced row echelon form of a matrix is unique.