

Elementary row operations:

Operation 1: switch 2 rows

E.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operation 2: scale one row by a ^{nonzero} real/complex number

E.g. $\begin{bmatrix} 1+i & 1-i \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \times (2+2i) \rightsquigarrow \begin{bmatrix} (1+i)(2+2i) & (1-i)(2+2i) \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 4i & 4 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$\begin{bmatrix} 2+2i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 1-i \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Operation 3: add a \mathbb{R}/\mathbb{C} -multiple of one row to another

E.g. $\begin{bmatrix} 1+i & 2 \\ 3 & 4 \\ 4 & 4 \end{bmatrix} + (-2+2i) \cdot \text{row 1} \rightsquigarrow \begin{bmatrix} 1+i & 2 \\ 3 & 4 \\ 4 + (-2+2i)(1+i) & 4 + (-2+2i)2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2+2i & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 2 \\ 3 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1+i & 2 \\ 3 & 4 \\ 0 & 4i \end{bmatrix}$$

Elementary column operations:

Operation 1: switch 2 columns

E.g. $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 5 & 3 & 1 \\ 6 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Operation 2: scale one column by a ^{nonzero} real/complex number

E.g. $\begin{bmatrix} 1+i & 3 & 5 \\ 1-i & 4 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} (1+i)(2+2i) & 3 & 5 \\ (1-i)(2+2i) & 4 & 6 \end{bmatrix} = \begin{bmatrix} 4i & 3 & 5 \\ 4 & 4 & 6 \end{bmatrix}$

$\times (2+2i)$

$$\begin{bmatrix} 1+i & 3 & 5 \\ 1-i & 4 & 6 \end{bmatrix} \parallel \begin{bmatrix} 2+2i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

Operation 3: add a ^{nonzero} \mathbb{R}/\mathbb{C} -multiple of one column to another

E.g. $\begin{bmatrix} 1+i & 3 & 4 \\ 2 & 4 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1+i & 3 & 4 + (-2+2i)(1+i) \\ 2 & 4 & 4 + (-2+2i)2 \end{bmatrix} = \begin{bmatrix} 1+i & 3 & 0 \\ 2 & 4 & 4i \end{bmatrix}$

$+ (-2+2i) \cdot \text{column 1}$

$$\begin{bmatrix} 1+i & 3 & 4 \\ 2 & 4 & 4 \end{bmatrix} \parallel \begin{bmatrix} 1 & 0 & -2+2i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

Def. The **matrices** by which multiplications give elementary row/column operations are called **elementary matrices**

* Note: Elementary matrices are exactly the matrices you get by performing an elementary row/column operation to I_n

E.g. Let $T: \mathbb{R}^5 \rightarrow \mathbb{R}^3$ be given by the matrix

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix}$$

under standard bases. Compute $K(T), R(T)$.

$\mathcal{X} = \{ (1, 0, 0), (0, 1, 0), (0, 0, 1) \}$ standard basis for \mathbb{R}^3
 $\mathcal{Y} = \{ (1, 0, 0, 0, 0), \dots, (0, 0, 0, 0, 1) \}$ standard basis for \mathbb{R}^5

$[T]_{\mathcal{X}\mathcal{Y}} v = (x_1, x_2, x_3, x_4, x_5)$ in \mathbb{R}^5 , $T(v) =$

$$\begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{cases} y_1 \cdot (1, 0, 0) + \\ y_2 \cdot (0, 1, 0) + \\ y_3 \cdot (0, 0, 1) \end{cases} = (y_1, y_2, y_3) \text{ in } \mathbb{R}^3$$

$K(T)$ = the set of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ in \mathbb{R}^5 such that

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ i.e., } K(T) \text{ is given by the}$$

set of real solutions to
$$\begin{cases} 1x_1 + (-2)x_2 + 0x_3 + 2x_4 + (-3)x_5 = 0 \\ 2x_1 + (-4)x_2 + 2x_3 + 0x_4 + 8x_5 = 0 \\ 1x_1 + (-2)x_2 + 3x_3 + (-3)x_4 + 16x_5 = 0 \end{cases}$$

The 3 elementary row operations on the matrix

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix}$$

correspond to switching 2 equations, scaling one equation by a real number, and adding a real-multiple of one

equation to another. Therefore, they do not change the set of solutions.

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 2 & -4 & 2 & 0 & 8 \\ 1 & -2 & 3 & -3 & 16 \end{bmatrix} \begin{matrix} \\ + (-2) \cdot \text{row 1} \\ + (-1) \cdot \text{row 1} \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 2 & -4 & 14 \\ 0 & 0 & 3 & -5 & 19 \end{bmatrix} \times \frac{1}{2}$$

this process is called Gaussian Elimination

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 & 7 \\ 0 & 0 & 3 & -5 & 19 \end{bmatrix} + (-3) \cdot \text{row 2}$$

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{array}{l} + (-2) \cdot \text{row 3} \\ + 2 \cdot \text{row 3} \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \leftarrow \text{reduced row echelon form of the matrix}$$

x_2 and x_5 are free variables

can take any real value

$$1x_4 + (-2)x_5 = 0 \Rightarrow x_4 = 2x_5$$

$$1x_3 + 3x_5 = 0 \Rightarrow x_3 = -3x_5$$

$$1x_1 + (-2)x_2 + 1x_5 = 0 \Rightarrow x_1 = 2x_2 - x_5$$

x_2, x_5 any real value

general solution.

$$K(T) = \left\{ \begin{bmatrix} 2x_2 - x_5 \\ x_2 \\ -3x_5 \\ 2x_5 \\ x_5 \end{bmatrix}, x_2, x_5 \in \mathbb{R} \right\}$$

* Note that $R(T)$ is exactly the span of the column vectors of the matrix.

Dimension Theorem: V, W vector spaces over \mathbb{R}/\mathbb{C} , $T: V \rightarrow W$ linear transformation, then $\dim V = \underbrace{\dim K(T)}_{\text{nullity of } T} + \underbrace{\dim R(T)}_{\text{rank of } T}$

rank of the matrix =

dimension of
column span
of the matrix
||
dimension of
row span of
the matrix

Definitions A matrix is said to be in **row echelon form** if it satisfies the following three conditions:

1. Each nonzero row lies above every zero row.
2. The leading entry of a nonzero row lies in a column to the right of the column containing the leading entry of any preceding row.
3. If a column contains the leading entry of some row, then all entries of that column below the leading entry are 0.⁵

If a matrix also satisfies the following two additional conditions, we say that it is in **reduced row echelon form**.⁶

4. If a column contains the leading entry of some row, then all the other entries of that column are 0.
 5. The leading entry of each nonzero row is 1.
-

* It can be shown that the reduced row echelon form of a matrix is unique.