$$\{a_0+a_1x+a_2x^2 \mid a_0,a_1,a_2 \text{ real numbers}\} \qquad \{a_0+a_1x \mid a_0,a_1 \text{ real numbers}\}$$

E.g. (Workshop 6 Problem 2) $T: P_2 \longrightarrow P_1$

T = Differentiation; bases: $\mathcal{Y} = \{1, x, x^2\}$, $\mathcal{X} = \{1, x\}$

$$\begin{cases} T(1) = 0 = 0\cdot 1 + 0\cdot x \\ T(x) = 1 = 1\cdot 1 + 0\cdot x \\ T(x^2) = 2x = 0\cdot 1 + 2\cdot x \end{cases} \Rightarrow [T]_{\mathcal{X}\mathcal{Y}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$T(3+2x+x^2) = \ ?$$

$$= T(3\cdot 1 + 2x + 1\cdot x^2) = 3T(1) + 2T(x) + 1T(x^2)$$

T preserves linear combinations!

$$= 3(0\cdot 1 + 0\cdot x) + 2(1\cdot 1 + 0\cdot x) + 1(0\cdot 1 + 2\cdot x)$$

$$= (3\cdot 0 + 2\cdot 1 + 1\cdot 0)\cdot 1 + (3\cdot 0 + 2\cdot 0 + 1\cdot 2)\cdot x$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3\cdot 0 + 2\cdot 1 + 1\cdot 0 \\ 3\cdot 0 + 2\cdot 0 + 1\cdot 2 \end{bmatrix} !!$$

$$\int_0^1 0\,dx = 0$$

$$\int_0^2 0\,dx = 0$$

- Vector space over $\mathbb{R}/\mathbb{C}$
- Linear transformation
- Composition of linear transformations
- Invertibility of linear transformations

$\xrightarrow[\text{bases}]{\text{choose}}$

- $\mathbb{R}^n/\mathbb{C}^n$
- Matrices with entries in $\mathbb{R}/\mathbb{C}$
- Product of matrices
- Invertibility of matrices

$$\begin{tikzcd} U \arrow[r, "S"] \arrow[rr, bend right, "T \circ S"'] & V \arrow[r, "T"] & W \end{tikzcd}$$

$\mathcal{X}$ $\qquad$ $\mathcal{Y}$ $\qquad$ $\mathcal{Z}$ $\quad$ bases

$$[T \circ S]_{\mathcal{Z}\mathcal{X}} = [T]_{\mathcal{Z}\mathcal{Y}} \cdot [S]_{\mathcal{Y}\mathcal{X}}$$

(U, V are vector spaces over $\mathbb{R}/\mathbb{C}$.)

Def. A linear transformation $S : U \to V$ (bases $\mathcal{X}$, $\mathcal{Y}$) is called **invertible** if there is a linear transformation $T : V \to U$ such that $S \circ T = id_V$, $T \circ S = id_U$. In this case, T is called the **inverse** of S (if exists it is unique).

Fact (Workshop 6 Problem 3):

S is invertible $\Longleftrightarrow$ $[S]_{\mathcal{Y}\mathcal{X}}$ is invertible

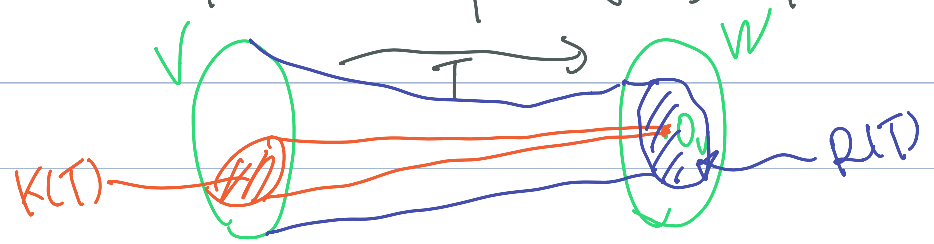
T is the inverse of S $\Longleftrightarrow$ $[T]_{\mathcal{X}\mathcal{Y}} = [S]_{\mathcal{Y}\mathcal{X}}^{-1}$

Def. (Workshop 6 Problem 4) Let V, W be vector spaces over $\mathbb{R}/\mathbb{C}$ and $T : V \to W$ be a linear transformation.

We define the **kernel** of T (denoted **$K(T)$**) to be the set of v in V such that $T(v) = 0_W$.

We define the **range** of T (denoted **$R(T)$**) to be the set of w in W such that there exists a v in V with $T(v) = w$.

Check: $K(T)$ is a subspace of V; $R(T)$ is a subspace of W.



$\{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \text{ real numbers}\}$ $\{a_0 + a_1 x \mid a_0, a_1 \text{ real numbers}\}$

E.g. (Workshop 6 Problem 2) $\underset{\text{Differentiation}}{\underset{\parallel}{T}} : P_2 \longrightarrow P_1$

Find a basis for $K(T)$ and a basis for $R(T)$.

$K(T) = P_0$: $\{1\}$ ✓

$R(T) = P_1$: $\{1, x\}$.