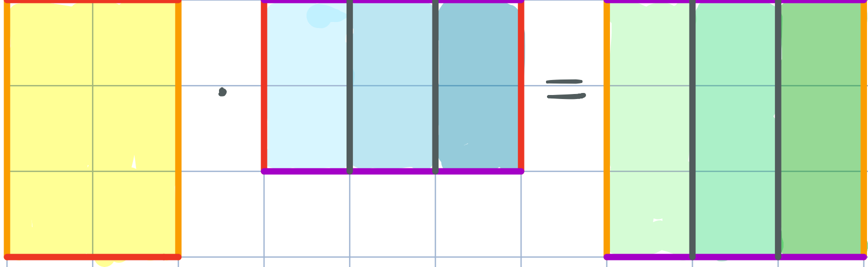

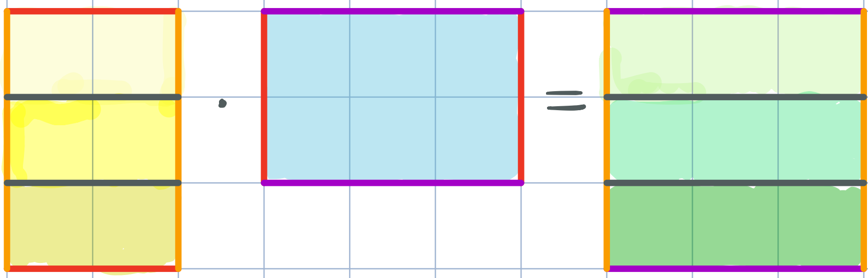

More generally:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix}$$

$$C_{11} = A_{11}B_{11} + A_{12}B_{21} \qquad C_{12} = A_{11}B_{12} + A_{12}B_{22}$$

$$C_{21} = A_{21}B_{11} + A_{22}B_{21} \qquad C_{22} = A_{21}B_{12} + A_{22}B_{22}$$

$$C_{31} = A_{31}B_{11} + A_{32}B_{21} \qquad C_{32} = A_{31}B_{12} + A_{32}B_{22}$$

E.g.

$$\begin{bmatrix} 0 & A \\ I_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} B & C & D \\ E & 0 & F \end{bmatrix} = \begin{bmatrix} AE & 0 & AF \\ B & C & D \end{bmatrix}.$$

this means for each $v$ in $V$ we assign a unique $T(v)$ in $W$

**Def.** Let $V, W$ be vector spaces over $\mathbb{R}/\mathbb{C}$, a map $T: V \to W$ is called a linear transformation if for any $v, v'$ in $V$ and $\lambda$ in $\mathbb{R}/\mathbb{C}$:

- Scalar multiplications are preserved: $T(\lambda \cdot v) = \lambda \cdot T(v)$
- Additions are preserved: $T(v+v') = T(v) + T(v')$

as vectors in $W$.

☆ The 2 conditions are equivalent to the condition that linear combinations are preserved.

**E.g.s** ?

Trivial ones:

vector space of linear transformations from $V$ to $W$

1. Zero map $0$: send everything to the zero vector in $W$
2. Identity map $id: V \to V$: does nothing, $id(v) = v$ for all $v$

sometimes also write $id_V$ to indicate it's the identity map on $V$.

Interesting ones:

1. Differentiation:

{differentiable $\mathbb{R}$-valued functions} $\to$ {$\mathbb{R}$-valued functions}

2. $\int_a^x$:

some fixed real number

{integrable $\mathbb{R}$-valued functions} $\to$ {differentiable $\mathbb{R}$-valued functions}

$$f(x) \longmapsto \int_a^x f(t)\,dt$$

☆ When $T: V \to W$ is a linear transformation, we have:

1. $T(0_V) = 0_W$ $\quad (T(0_V) = T(0 \cdot 0_V) = 0 \cdot T(0_V))$

in $\mathbb{R}/\mathbb{C}$

2. $T(-v) = -T(v)$

$(T(-v) = T((-1)v) = (-1) \cdot T(v) = -T(v))$

in $\mathbb{R}/\mathbb{C}$

Exercise: A linear transformation $V \to W$ is determined by what it does on any subset $S$ of $V$ that spans $V$.

$S = \{v_1, v_2\}$, $\operatorname{span} S = V$, $T: V \to W$, $T(v_1), T(v_2)$ known

$v = a_1 v_1 + a_2 v_2$

$T(v) = T(a_1 v_1 + a_2 v_2) = a_1 T(v_1) + a_2 T(v_2)$ $T(v)$ is determined for all $v$ in $V$.

E.g. Let $V$ be 2-dimensional vector space with basis $\{v_1, v_2\}$, $W$ be a 3-dimensional vector space with basis $\{w_1, w_2, w_3\}$, then a linear transformation $T: V \to W$ is determined by $T(v_1), T(v_2)$, but

$T(v_1) = a_{11} w_1 + a_{21} w_2 + a_{31} w_3$ for some $a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32}$ in $\mathbb{R}$

$T(v_2) = a_{12} w_1 + a_{22} w_2 + a_{32} w_3$

In other words, $T$ is represented by the matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$

In general, when $V$ is an $m$-dimensional vector space with basis $\mathcal{X}$, $W$ is an $n$-dimensional vector space with basis $\mathcal{Y}$, a linear transformation $T: V \to W$ is represented by an $n \times m$ matrix $[T]_{\mathcal{Y}\mathcal{X}}$ (depends on the choice of bases)