Def. A basis for a vector space V is a linearly independent (1) subset of V whose span is V. (2)

* There are usually infinitely many bases for one vector space, but they all have the same size, which is called the dimension of the vector space.
* There are vector spaces with infinite dimension

E.g. the set of all polynomials with the usual $+, \cdot$

a basis: $\{1, x, x^2, \dots\}$

but we will only work with finite dimensional vector spaces in this class.

Observation: if $\{e_1, \cdots, e_n\}$ is a basis for a vector space V, then every v in V is equal to $a_1 \cdot e_1 + \cdots + a_n \cdot e_n$ for unique $a_1, \cdots, a_n$ in $\mathbb{R}/\mathbb{C}$. Therefore, by choosing a basis, every vector space of dimension n can be identified with $\mathbb{R}^n/\mathbb{C}^n$.

$$v = a_1 \cdot e_1 + \cdots + a_n \cdot e_n = b_1 \cdot e_1 + \cdots + b_n \cdot e_n$$

$$(a_1 - b_1) \cdot e_1 + \cdots + (a_n - b_n) \cdot e_n = 0 \quad *$$

$$a_1 - b_1 = 0, \dots, (a_n - b_n) = 0$$

$$a_1 = b_1, \dots, a_n = b_n.$$

E.g. (Also see Homework 1 Problem 4) The set of real-valued solutions to the differential equation $y'' + y' + y = 0$ is a vector space over $\mathbb{R}$ of dimension 2. A basis $\{y_1, y_2\}$

for this vector space is called a fundamental set of solutions. Every real-valued solution can be written as $a_1 y_1 + a_2 y_2$ for unique a_1, a_2 in $\mathbb{R}$. (Wronskian is coming up.)

E.g. (See Homework 1 Problem 3) Find a basis for $\underline{Mat_{2\times 3}}(\mathbb{R})$. What's the dimension of $Mat_{2\times3}(\mathbb{R})$?

$$= 6$$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = a \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

☆ Connection among linear (in)dependence, dimension and span: a set of vectors in a vector space is linearly independent if the dimension of the span equals the number of elements in the set. if the dimension is smaller, then the set is linearly dependent.