Observation: the span of a nonempty subset S of V consists exactly of all vectors of the form $a_1 v_1 + \cdots + a_k v_k$, where k is some natural number, $a_1, \ldots, a_k$ are in $\mathbb{R}/\mathbb{C}$, and $v_1, \ldots, v_k$ are in S

a vector v in S is redundant

= removing it doesn't change the span

= it's in the span of the rest of the vectors

= there exist $a_1, \ldots, a_k$ in $\mathbb{R}/\mathbb{C}$ and $v_1, \ldots, v_k$ in S such that $v = a_1 v_1 + \cdots + a_k v_k$ $\qquad \neq v$

at least one vector in S is redundant

= there exist $v, v_1, \ldots, v_k$ distinct elements of S such that $v = a_1 v_1 + \cdots + a_k v_k$ for some $a_1, \ldots, a_k$ in $\mathbb{R}/\mathbb{C}$

= there exist $v_1, \ldots, v_l$ distinct elements of S and $b_1, \ldots, b_l$ in $\mathbb{R}/\mathbb{C}$ not all zero, such that $b_1 v_1 + \cdots + b_l v_l = 0$

say $b_1 \neq 0$, then

$$v_1 + \frac{b_2}{b_1} v_2 + \cdots + \frac{b_l}{b_1} v_l = 0$$

$$v_1 = -\frac{b_2}{b_1} v_2 - \cdots - \frac{b_l}{b_1} v_l$$

= S is linearly dependent

no vector in S is redundant

= the only times $b_1 v_1 + \cdots + b_l v_l = 0$ for a positive integer l, real/complex numbers $b_1, \ldots, b_l$, and $v_1, \ldots, v_l$ distinct elements of S is when $b_1 = \cdots = b_l = 0$

= S is linearly independent

Def. A basis for a vector space V is a linearly independent subset of V whose span is V.

* There are usually infinitely many bases for one vector space, but they all have the same size, which is called the dimension of the vector space.
* There are vector spaces with infinite dimension

E.g. the set of all polynomials with the usual $+, \cdot$

a basis: $\{1, x, x^2, \dots\}$

but we will only work with finite dimensional vector spaces in this class.

Observation: if $\{e_1, \dots, e_n\}$ is a basis for a vector space V, then every v in V is equal to $a_1 \cdot e_1 + \dots + a_n \cdot e_n$ for unique $a_1, \dots, a_n$ in $\mathbb{R}/\mathbb{C}$. Therefore, by choosing a basis, every vector space of dimension n can be identified with $\mathbb{R}^n$.