Eg. $\begin{cases} x+y+z=0 \\ x+y-z=0 \end{cases}$ (1) $\Longleftrightarrow$ $\begin{cases} x+y=0 \\ z=0 \end{cases}$

What are some real solutions?

$\begin{cases} x=0 \\ y=0 \\ z=0 \end{cases}$ $\begin{cases} x=1 \\ y=-1 \\ z=0 \end{cases}$ $\begin{cases} x=-\sqrt{2} \\ y=\sqrt{2} \\ z=0 \end{cases}$ ...

$-3\cdot$ $\Downarrow$ $+$ $\Downarrow$

$\begin{cases} x=-3 \\ y=3 \\ z=0 \end{cases}$ $\begin{cases} x=1-\sqrt{2} \\ y=\sqrt{2}-1 \\ z=0 \end{cases}$

The set of real \ complex solutions to (1) with $+$ (addition) and $\cdot$ (scalar multiplication) as defined, is our 1st example of a vector space over $\mathbb{R}$ \ $\mathbb{C}$.

Def. A vector space over $\mathbb{R}$ is a set V with 2 operations called $+$ (addition) and $\cdot$ (scalar multiplication) defined so that for any element u, v in V, $u+v$ is a unique element in V, for any a in $\mathbb{R}$ and v in V, $a \cdot v$ is a unique element in V, and they satisfy the following axioms:

$+$:

1. $u+v=v+u$
2. $(u+v)+w=u+(v+w)$

$\Big\}$ for all u,v,w in V

3. There is a 0 (zero element) in V such that $0+v=v$ for all v in V ! Not to be confused with 0 in $\mathbb{R}$
4. For every v in V there is a $-v$ in V such that $v+(-v)=0$

$\cdot$:

1. $1\cdot v=v$
2. $(ab)\cdot v=a\cdot(b\cdot v)$
3. $a\cdot(u+v)=a\cdot u+a\cdot v$
4. $(a+b)\cdot v=a\cdot v+b\cdot v$

$\Bigg\}$ for all a,b in $\mathbb{R}$ and u,v in V

* Elements of a vector space are also called vectors
* If we replace $\mathbb{R}$ with $\mathbb{C}$ everywhere then it's a vector space over $\mathbb{C}$