Quantum K-theory of Incidence Varieties

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Summary

We prove a conjecture of Buch and Mihalcea in the case of the incidence variety $X = \mathrm{Fl}(1, n-1; n)$ and determine the structure of its (T-equivariant) quantum K-theory ring. Our results are an interplay between geometry and combinatorics. The geometric side concerns Gromov-Witten varieties of 3-pointed genus 0 stable maps to X with markings sent to Schubert varieties, while on the combinatorial side are formulas for the (equivariant) quantum K-theory ring of X. We prove that the Gromov-Witten variety is rationally connected when one of the defining Schubert varieties is a divisor and another is a point. This implies that the (equivariant) K-theoretic Gromov-Witten invariants defined by two Schubert classes and a Schubert divisor class can be computed in the ordinary (equivariant) K-theory ring of X. We derive a positive Chevalley formula for the equivariant quantum K-theory ring of X and a positive Little-Richardson rule for the non-equivariant quantum K-theory ring of X. The Littlewood-Richardson rule in turn implies that non-empty Gromov-Witten varieties given by Schubert varieties in general position have arithmetic genus 0.

The Incidence Variety

$$
\begin{aligned}
X &= \mathrm{Fl}(1, n-1; n) = \mathrm{SL}(\mathbb{C}^n)/P \\
&= \{U \subset V \subset \mathbb{C}^n : \dim U = 1, \dim V = n-1\} \\
&= \{x_1 y_1 + \cdots + x_n y_n = 0\} \subset \mathbb{P}(\mathbb{C}^n) \times \mathbb{P}(\mathbb{C}^{n*})
\end{aligned}
$$

Schubert varieties in X are indexed by

$$W^P := \{[i,j] : 1 \le i \ne j \le n\}.$$

$X_{[i,j]} = \{x_{i+1} = \cdots = x_n = y_1 = \cdots = y_{j-1} = 0\} \subseteq X,$
$X^{[i,j]} = \{x_1 = \cdots = x_{i-1} = y_{j+1} = \cdots = y_n = 0\} \subseteq X,$
$D^{[1]} := X^{[2,n]} = \{x_1 = 0\},\ D^{[2]} := X^{[1,n-1]} = \{y_n = 0\}.$

$\overline{M}_{0,3}(X, d)$ and Related Constructions

Fix $d \in H_2(X)^+ = \mathbb{Z}^2_{\ge 0}$.
$M_d := \overline{M}_{0,3}(X, d) := \overline{\{f : \mathbb{P}^1 \to X \mid f_*[\mathbb{P}^1] = d\}}$.
$\mathrm{ev}_1,\ \mathrm{ev}_2,\ \mathrm{ev}_3 : M_d \to X$ (evaluate at markings $0,\ 1,\ \infty \in \mathbb{P}^1$)

For $u,\ v,\ w \in W^P$,
$M_d(X_u, X^v) := \mathrm{ev}_1^{-1}(X_u) \cap \mathrm{ev}_2^{-1}(X^v) \subseteq M_d$,
$\Gamma_d(X_u, X^v) := \mathrm{ev}_3(M_d(X_u, X^v)) \subseteq X$,
$\Gamma_d(X_u) := \mathrm{ev}_2(\mathrm{ev}_1^{-1}(X_u))$ is again a Schubert variety,

$$M_d(X_u, X^v, g.X^w) := \mathrm{ev}_1^{-1}(X_u) \cap \mathrm{ev}_2^{-1}(X^v) \cap \mathrm{ev}_3^{-1}(g.X^w).$$

Assume $g \in \mathrm{SL}(\mathbb{C}^n)$ general, then
the **cohomological Gromov-Witten Invariant**

$$I_d([X_u], [X^v], [X^w]) = \# M_d(X_u, X^v, g.X^w);$$

the **K-theoretic Gromov-Witten Invariant**

$$
\begin{aligned}
I_d([\mathcal{O}_{X_u}], [\mathcal{O}_{X^v}], [\mathcal{O}_{X^w}]) &= \chi(\mathcal{O}_{M_d(X_u, X^v, g.X^w)}) \\
&= \chi_{M_d}(\mathrm{ev}_1^*[\mathcal{O}_{X_u}] \cdot \mathrm{ev}_2^*[\mathcal{O}_{X^v}] \cdot \mathrm{ev}_3^*[\mathcal{O}_{X^w}]) \in K(\mathrm{pt});
\end{aligned}
$$

the **T-equivariant K-theoretic Gromov-Witten Invariant**

$$
\begin{aligned}
I_d^T([\mathcal{O}_{X_u}], [\mathcal{O}_{X^v}], [\mathcal{O}_{X^w}]) \\
= \chi_{M_d}^T(\mathrm{ev}_1^*[\mathcal{O}_{X_u}] \cdot \mathrm{ev}_2^*[\mathcal{O}_{X^v}] \cdot \mathrm{ev}_3^*[\mathcal{O}_{X^w}]) \in K^T(\mathrm{pt}),
\end{aligned}
$$

where $T \subset \mathrm{SL}(\mathbb{C}^n)$ is the maximal torus of diagonal matrices.
$K^T(\mathrm{pt})$ can be identified with the representation ring of T.
$[\mathcal{O}_{X_u}]$ and $[\mathcal{O}_{X^v}]$ form $K^T(\mathrm{pt})$-bases for $K_T(X)$.

X can be replaced by any flag variety G/P.

Results Part 1

Theorem 1 (X '21). *The general fibre of*

$$\mathrm{ev}_3 : M_d(X_{[i,j]}, D^{[k]}) \to \Gamma_d(X_{[i,j]}, D^{[k]})$$

is rationally connected.

Using a result of Kollár, this implies that

$$\mathrm{ev}_{3*}[\mathcal{O}_{M_d(X_{[i,j]}, D^{[k]})}] = [\mathcal{O}_{\Gamma_d(X_{[i,j]}, D^{[k]})}]$$

in (equivariant) K-theory, proving a conjecture of Buch and Mihalcea for X.

In the following, $\mathcal{O}_{[i,j]} := [\mathcal{O}_{X_{[i,j]}}],\ \mathcal{O}^{[k]} := [\mathcal{O}_{D^{[k]}}] \ \in K_T(X)$.

Corollary: "quantum equals classical" formula (X '21).

$$
\begin{aligned}
I_d^T(\mathcal{O}_{[i,j]}, \mathcal{O}^{[k]}, \sigma) &= \chi_X^T([\mathcal{O}_{\Gamma_d(X_{[i,j]}, D^{[k]})}] \cdot \sigma) \\
&= \begin{cases} \chi_X^T([\mathcal{O}_{\Gamma_d(X_{[i,j]})}] \cdot \sigma) & \text{if } d_k > 0 \\ \chi_X^T([\mathcal{O}_{\Gamma_d(X_{[i,j]})}] \cdot \mathcal{O}^{[k]} \cdot \sigma) & \text{if } d_k = 0 \end{cases}.
\end{aligned}
$$

The right hand side is easily computable using Lenart and Postnikov's Chevalley formula for $K_T(X)$.

Quantum K-theory

Quantum K-theory was introduced by Givental and Lee as a K-theoretic analogue of quantum cohomology.
The (small) T-equivariant quantum K-theory ring of X is an algebra $QK_T(X)$ over $K^T(\mathrm{pt})[[q_1, q_2]]$ with a $K^T(\mathrm{pt})[[q_1, q_2]]$-basis consisting of $\mathcal{O}^w$ for $w \in W^P$. Multiplication in $QK_T(X)$ is defined using $I_d^T(\sigma_1, \sigma_2, \sigma_3)$.
$\widetilde{W^P} := \{[i,j] \in \mathbb{Z} \times \mathbb{Z} : i \not\equiv j \bmod n\}$. For $w \in \widetilde{W^P}$,

$$\overline{w} := [\overline{i}, \overline{j}] \in W^P \text{ is defined by } \overline{i} \equiv i,\ \overline{j} \equiv j,$$

$$d(w) := \left(\frac{i - \overline{i}}{n}, \frac{\overline{j} - j}{n}\right),$$

$$\mathcal{O}^w := q^{d(w)} [\mathcal{O}_{X^{\overline{w}}}] \in QK_T(X)_q := QK_T(X) \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q, q^{-1}].$$

We write $\mathbb{C}_{\varepsilon_i}$ for the 1-dimensional representation of T given by the character ε_i which records the i-th diagonal entry.
We let $\varepsilon_i := \varepsilon_{\overline{i}}$ for $i \in \mathbb{Z}$.

Results Part 2

Equivariant Chevalley Formula (X '21).
In $QK_T(X)_q$, *for* $[i,j] \in \widetilde{W^P}$, $\mathcal{O}^{[i,j]} \star \mathcal{O}^{[1]}$ *equals*

$$(1 - [\mathbb{C}_{\varepsilon_i - \varepsilon_1}])\mathcal{O}^{[i,j]} + [\mathbb{C}_{\varepsilon_i - \varepsilon_1}]\mathcal{O}^{[i+1,j]} \text{ when } i+1 \not\equiv j \bmod n,$$

$$(1 - [\mathbb{C}_{\varepsilon_i - \varepsilon_1}])\mathcal{O}^{[i,j]} + [\mathbb{C}_{\varepsilon_i - \varepsilon_1}](\mathcal{O}^{[i+1,j-1]} + \mathcal{O}^{[i+2,j]} - \mathcal{O}^{[i+2,j-1]})$$

when $i + 1 \equiv j \bmod n$.
The formula for $\mathcal{O}^{[i,j]} \star \mathcal{O}^{[2]}$ *is analogous.*

The non-equivariant case ($[\mathbb{C}_\varepsilon] = 1$) is in Rosset's thesis ('20).
A formula for two-step varieties is in Kouno-Lenart-Naito-Sagaki ('21).

Non-equivariant Little-Richardson Rule (X '21).
In $QK(X)$, *for* $[i,j],\ [k,l] \in W^P$, $\mathcal{O}^{[i,j]} \star \mathcal{O}^{[k,l]}$ *equals*

$$\mathcal{O}^{[x,y]} \text{ when } x - y < n[\chi(i > j) + \chi(k > l)],$$

$$\mathcal{O}^{[x,y-1]} + \mathcal{O}^{[x+1,y]} - \mathcal{O}^{[x+1,y-1]} \text{ otherwise,}$$

where $x = i + k - 1,\ y = j + l - n$.

Corollary 2 (X '21). *For* $d \in H_2(X)^+$, *a general* $g \in \mathrm{SL}(\mathbb{C}^n)$, *and* $u,\ v,\ w \in W^P$, $M_d(g.X^u, X^v, X_w)$ *has arithmetic genus* 0 *whenever it is non-empty.*

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