Instructions

1. The statements in Italics are for introducing results and notations that may be used again in this course. You are only required to read and think about them.

2. To receive full credit you must explain how you got your answer.

3. While I encourage collaboration, you must write solutions IN YOUR OWN WORDS. DO NOT SHARE COMPLETE SOLUTIONS before they are due. YOU WILL RECEIVE NO CREDIT if you are found to have copied from whatever source or let others copy your solutions.

4. Homework must be handwritten (electronic handwriting is allowed) for authentication purposes and submitted on Canvas. Please do NOT include any personal information such as your name and netID in your file. Late homework will NOT be accepted. It is your responsibility to MAKE SURE THAT YOUR SUBMISSIONS ARE SUCCESSFUL AND YOUR FILES ARE LEGIBLE AND COMPLETE. It is also your responsibility that whoever reads your work will understand and enjoy it. Up to 4 points out of 40 may be taken off if your solutions are hard to read or poorly presented.

Homework 6 Solution

1. Find the least squares solution $\boldsymbol{x_0}$ to $A\boldsymbol{x} = \boldsymbol{b}$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$. Then check that $A\boldsymbol{x_0} = \boldsymbol{w}$ for the $\boldsymbol{w}$ you found in Workshop 16 Problem 2b (6 pts).

We solve $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$ (3 pts) (by Gaussian Elimination, for example, details omitted) to get $\boldsymbol{x_0} = \begin{bmatrix} 3/2 \\ 5/6 \end{bmatrix}$ (2 pts). Direct computation verifies $A\boldsymbol{x_0} = \begin{bmatrix} 5/6 \\ 7/3 \\ 23/6 \end{bmatrix} = \boldsymbol{w}$. (1 pt)

2. Suppose that a spring whose natural length is L inches is attached to a wall. A force y is applied to the free end of the spring, stretching the spring s inches beyond its natural length. Hooke's law states (within certain limits) that $y = ks$, where k is a constant called the spring constant. Now suppose that after the force y is applied, the new length of the spring is x. Then $s = x - L$, and Hooke's law yields $y = ks = k(x - L) = a + kx$, where $a = -kL$. Apply the method of least squares to the following data to estimate k and L (8 pts):

Length x in inches	Force y in pounds
3.5	1.0
4.0	2.2
4.5	2.8
5.0	4.3

The data gives us a system of equations $\begin{cases} 1.0 = a + k3.5 \\ 2.2 = a + k4.0 \\ 2.8 = a + k4.5 \\ 4.3 = a + k5.0 \end{cases}$ (2 pts). We can rewrite it as $A\boldsymbol{x} = \boldsymbol{b}$ for $A = \begin{bmatrix} 1 & 3.5 \\ 1 & 4.0 \\ 1 & 4.5 \\ 1 & 5.0 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} a \\ k \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 1.0 \\ 2.2 \\ 2.8 \\ 4.3 \end{bmatrix}$ (2 pts). One can verify (using Gaussian Elimination, for example) that this equation is inconsistent. Therefore, we solve $A^TA\boldsymbol{x} = A^T\boldsymbol{b}$ instead (1 pt) (details omitted) and get $\boldsymbol{x} = \begin{bmatrix} -6.35 \\ 2.1 \end{bmatrix}$ (1 pt). This means $a \approx -6.35$ pounds and $k \approx 2.1$ pounds per inch. Therefore $L = -a/k \approx 3.0238$ inch. (2 pts)

3. Use the method of least squares to find the parabola that best fits the data (9 pts):

$$(0, 2), (1, 2), (2, 4), (3, 8)$$

A parabola has equation $y = a + bx + cx^2$, so the data gives us the system of equations $\begin{cases} 2 = a + b0 + c0^2 \\ 2 = a + b1 + c1^2 \\ 4 = a + b2 + c2^2 \\ 8 = a + b3 + c3^2 \end{cases}$ (3 pts), which can be rewritten as $A\boldsymbol{x} = \boldsymbol{b}$ for $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 8 \end{bmatrix}$ (2 pts). One can verify (using Gaussian Elimination, for example) that this equation is inconsistent. Therefore, we solve $A^TA\boldsymbol{x} = A^T\boldsymbol{b}$ instead (1 pt) (details omitted) and get $\boldsymbol{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ (2 pts). Therefore, the parabola is $y = 2 - x + x^2$ (1 pt).

4. Determine whether this statement is true or false and explain why: for any inconsistent system of linear equations $A\mathbf{x} = \boldsymbol{b}$, the vector $\boldsymbol{x_0}$ for which $\|A\boldsymbol{x_0} - \boldsymbol{b}\|$ is a minimum is unique. (6 pts)

Let W be the span of the columns in A, then $\|A\boldsymbol{x_0} - \boldsymbol{b}\|$ is a minimum exactly when $A\boldsymbol{x_0} = U_W(\boldsymbol{b})$ or $A^TA\boldsymbol{x_0} = A^T\boldsymbol{b}$ (2 pts). The equation $A^TA\boldsymbol{x} = A^T\boldsymbol{b}$ has at least one solution by construction.

Suppose A is $m \times n$, then A^TA is $n \times n$. If A^TA is not invertible, or equivalently, if the rank of A^TA is smaller than n, then $A^TA\boldsymbol{x} = A^T\boldsymbol{b}$ will have infinitely many solutions (2 pts). In Workshop 17 problem 2b we saw that A^TA and A have the same rank. Therefore, $A^TA\boldsymbol{x} = A^T\boldsymbol{b}$ will have infinitely many solutions when the rank of A is smaller than n, or equivalently, when the columns in A are linearly dependent (2 pts). Therefore, the statement is false.

Alternatively, one could demonstrate a specific counterexample. (6 pts)

5. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$. Find an orthogonal matrix Q and diagonal matrix D such that $A = QDQ^T$. (11 pts)

For your entertainment, I did this problem in a nonstandard way below. Of course one could just compute eigenvalues and eigenspaces as usual and use Gram-Schmit with normalization to find an orthonormal basis for each eigenspace. 1 pt for characteristic polynomial, 2 pts for eigenvalues, 3 pts for eigenvectors, 2 pts for Gram-Schmit, 2 pts for normalization, 1 pt for final Q and D.

A short cut to finding an eigenvalue: notice (through Gaussian Elimination, for example) A has rank 1, so by Workshop 17 problem 1, 0 is an eigenvalue with multiplicity 2.

Let λ be the other eigenvalue. Since A is symmetric, eigenvectors with distinct eigenvalues are orthogonal to each other, so the $\lambda-$eigenspace must be the orthogonal complement of the $0-$eigenspace. This gives an alternative way to compute the $\lambda-$eigenspace.

By solving $(A - 0I)\boldsymbol{x} = \mathbf{0}$ (details omitted), we get a basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ for the $0-$eigenspace. We apply Gram-Schmit process with normalization to

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(Here the third vector was chosen to be outside of the span of the first two, and I'm using Gram-Schmit to find a basis for the orthogonal complement–why does it work?) to obtain an orthonormal basis $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$ for $\mathbb{R}^3$. By construction, this is an eigenbasis with the third vector $\boldsymbol{v}$ being an eigenvector with eigenvalue λ. (Why?) Because $A\boldsymbol{v} = 3\boldsymbol{v}$ by direct computation, we conclude that $\lambda = 3$. Hence, by the same process as before (see H4 problem 4b, for example), we have $Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Since Q is an orthogonal matrix by construction, $Q^{-1} = Q^T$.