Instructions

1. The statements in Italics are for introducing results and notations that may be used again in this course. You are only required to read and think about them.

2. To receive full credit you must explain how you got your answer.

3. While I encourage collaboration, you must write solutions IN YOUR OWN WORDS. DO NOT SHARE COMPLETE SOLUTIONS before they are due. YOU WILL RECEIVE NO CREDIT if you are found to have copied from whatever source or let others copy your solutions.

4. Homework must be handwritten (electronic handwriting is allowed) for authentication purposes and submitted on Canvas. Please do NOT include any personal information such as your name and netID in your file. Late homework will NOT be accepted. It is your responsibility to MAKE SURE THAT YOUR SUBMISSIONS ARE SUCCESSFUL AND YOUR FILES ARE LEGIBLE AND COMPLETE. It is also your responsibility that whoever reads your work will understand and enjoy it. Up to 4 points out of 40 may be taken off if your solutions are hard to read or poorly presented.

Homework 5 Solution

1. Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^m$ are vectors satisfying:

$$\|\boldsymbol{u}\| = 2 \quad \|\boldsymbol{v}\| = 3 \quad \|\boldsymbol{w}\| = 4 \quad \boldsymbol{u}\cdot\boldsymbol{v} = -1 \quad \boldsymbol{u}\cdot\boldsymbol{w} = 2 \quad \boldsymbol{v}\cdot\boldsymbol{w} = -2$$

Compute the following expressions: (2 pts each)

a. $(2\boldsymbol{u}+\boldsymbol{v})\cdot(3\boldsymbol{v}-4\boldsymbol{w})$
b. $\|\boldsymbol{u}+\boldsymbol{v}\|^2$
c. $\|-6\boldsymbol{w}\|$
d. $\|2\boldsymbol{v}-\boldsymbol{w}\|$

a. $(2\boldsymbol{u}+\boldsymbol{v})\cdot(3\boldsymbol{v}-4\boldsymbol{w}) = 6\boldsymbol{u}\cdot\boldsymbol{v} - 8\boldsymbol{u}\cdot\boldsymbol{w} + 3\boldsymbol{v}\cdot\boldsymbol{v} - 4\boldsymbol{v}\cdot\boldsymbol{w}$ (1 pt)$= 6\times(-1) - 8\times 2 + 3\times 3^2 - 4\times(-2) = 13$ (1 pt).

b. $\|\boldsymbol{u}+\boldsymbol{v}\|^2 = (\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{u}\cdot\boldsymbol{u} + 2\boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{v}$ (1 pt)$= 2^2 + 2\times(-1) + 3^2 = 11$ (1 pt).

c. $\|-6\boldsymbol{w}\| = \sqrt{(-6\boldsymbol{w})\cdot(-6\boldsymbol{w})} = 6\sqrt{\boldsymbol{w}\cdot\boldsymbol{w}} = 6\|\boldsymbol{w}\|$ (1 pt)$= 6\times 4 = 24$ (1 pt).

d. $\|2\boldsymbol{v}-\boldsymbol{w}\| = \sqrt{(2\boldsymbol{v}-\boldsymbol{w})\cdot(2\boldsymbol{v}-\boldsymbol{w})} = \sqrt{4\boldsymbol{v}\cdot\boldsymbol{v} - 4\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w}}$ (1 pt)$= \sqrt{4\times 3^2 - 4\times(-2) + 4^2} = 2\sqrt{15}$ (1 pt).

2. Use dot products to represent $\boldsymbol{u} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ as a linear combination of the vectors in the orthogonal set $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$. (4 pts)

Let $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{v_2} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \boldsymbol{v_3} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, then

$$\boldsymbol{u} = \frac{\boldsymbol{u} \cdot \boldsymbol{v_1}}{\boldsymbol{v_1} \cdot \boldsymbol{v_1}}\boldsymbol{v_1} + \frac{\boldsymbol{u} \cdot \boldsymbol{v_2}}{\boldsymbol{v_2} \cdot \boldsymbol{v_2}}\boldsymbol{v_2} + \frac{\boldsymbol{u} \cdot \boldsymbol{v_3}}{\boldsymbol{v_3} \cdot \boldsymbol{v_3}}\boldsymbol{v_3}$$

(numerator and denominator of each coefficient is worth 0.5 pt each)

$$= \frac{5}{2}\boldsymbol{v_1} + \frac{3}{6}\boldsymbol{v_2} + \frac{0}{3}\boldsymbol{v_3} = \frac{5}{2}\boldsymbol{v_1} + \frac{1}{2}\boldsymbol{v_2}$$

(1 pt for computation).

3. Find an orthogonal matrix with first column $\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$. (7 pts)

Let $\boldsymbol{u_1} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \boldsymbol{u_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{u_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{u_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, then $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ is linearly independent. We perform Gram-Schmit on this set with normalization. The resulting vectors will form the columns of an orthogonal matrix with the first column being $\boldsymbol{u_1}$.

$$\boldsymbol{v_1} = \boldsymbol{u_1} \text{ (already normal).}$$

$$\boldsymbol{v_2} = \boldsymbol{u_2} - (\boldsymbol{u_2} \cdot \boldsymbol{v_1})\boldsymbol{v_1} = \frac{1}{4}\begin{bmatrix} 3 \\ -1 \\ -1 \\ -1 \end{bmatrix}. \ \boldsymbol{v_2'} = \frac{\boldsymbol{v_2}}{\|\boldsymbol{v_2}\|} = \frac{1}{2\sqrt{3}}\begin{bmatrix} 3 \\ -1 \\ -1 \\ -1 \end{bmatrix}.$$

$$\boldsymbol{v_3} = \boldsymbol{u_3} - (\boldsymbol{u_3} \cdot \boldsymbol{v_1})\boldsymbol{v_1} - (\boldsymbol{u_3} \cdot \boldsymbol{v_2'})\boldsymbol{v_2'} = \frac{1}{3}\begin{bmatrix} 0 \\ 2 \\ -1 \\ -1 \end{bmatrix}. \ \boldsymbol{v_3'} = \frac{\boldsymbol{v_3}}{\|\boldsymbol{v_3}\|} = \frac{1}{\sqrt{6}}\begin{bmatrix} 0 \\ 2 \\ -1 \\ -1 \end{bmatrix}.$$

$$\boldsymbol{v_4} = \boldsymbol{u_4} - (\boldsymbol{u_4} \cdot \boldsymbol{v_1})\boldsymbol{v_1} - (\boldsymbol{u_4} \cdot \boldsymbol{v_2'})\boldsymbol{v_2'} - (\boldsymbol{u_4} \cdot \boldsymbol{v_3'})\boldsymbol{v_3'} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}. \ \boldsymbol{v_4'} = \frac{\boldsymbol{v_4}}{\|\boldsymbol{v_4}\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Therefore, $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{3} & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} \end{bmatrix}$ is one such matrix. (1 pt for finding independent set of vectors; 3 pts for Gram-Schmit; 3 pts for normalization.)

4. When do we have $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$ for vectors $\boldsymbol{u}, \boldsymbol{v}$ in $\mathbb{R}^n$? Explain why. (Hint: Think geometrically.) (4 pts)

Argument 1:

When $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent, $\|\boldsymbol{u}+\boldsymbol{v}\| < \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$, because the length of one side of a triangle is less than the sum of the lengths of the other two sides. (2 pts)

When $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly dependent, we have $\boldsymbol{u} = a\boldsymbol{v}$ for some a in $\mathbb{R}$ or $\boldsymbol{v} = b\boldsymbol{u}$ for some b in $\mathbb{R}$. Suppose $\boldsymbol{u} = a\boldsymbol{v}$ for some a in $\mathbb{R}$. Then $\|\boldsymbol{u}+\boldsymbol{v}\| = \|a\boldsymbol{v}+\boldsymbol{v}\| = |a+1|\|\boldsymbol{v}\|$, $\|\boldsymbol{u}\|+\|\boldsymbol{v}\| = \|a\boldsymbol{u}\|+\|\boldsymbol{v}\| = (|a|+1)\|\boldsymbol{v}\|$. Therefore, when $|a+1| = |a|+1$ or $\|\boldsymbol{v}\|$, that is, when $a \geq 0$ or $\boldsymbol{v} = \boldsymbol{0}$, we have $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$. Similarly, when $\boldsymbol{u} = \boldsymbol{0}$ or $\boldsymbol{v} = b\boldsymbol{u}$ for some non-negative b in $\mathbb{R}$, we have $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$. In summary, $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$ exactly when $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly dependent (1 pt) and $\boldsymbol{u}\cdot\boldsymbol{v} \geq 0$ (1 pt).

Argument 2:

Since both sides are non-negative, $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$ is equivalent to $\|\boldsymbol{u}+\boldsymbol{v}\|^2 = (\|\boldsymbol{u}\|+\|\boldsymbol{v}\|)^2$ (1 pt), or $(\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{u}\cdot\boldsymbol{u}+2\|\boldsymbol{u}\|\|\boldsymbol{v}\|+\boldsymbol{v}\cdot\boldsymbol{v}$, i.e., $\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|$. *Note that Cauchy-Schwartz inequality says* $\boldsymbol{u}\cdot\boldsymbol{v} \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$. When $\boldsymbol{u} = \boldsymbol{0}$ or $\boldsymbol{v} = \boldsymbol{0}$, we have $\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\| = 0$; when $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero, $\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos\theta$, where θ is the angle from $\boldsymbol{u}$ to $\boldsymbol{v}$ (1 pt). Therefore, the equality holds exactly when $\boldsymbol{u} = \boldsymbol{0}$ or $\boldsymbol{v} = \boldsymbol{0}$ or $\cos\theta = 1$ (2 pts).

5. Finish Workshop 16 Problem 2b. (5 pts)

In part a we found $\boldsymbol{w_1} = \frac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\2\end{bmatrix}, \boldsymbol{w_2} = \frac{1}{\sqrt{30}}\begin{bmatrix}5\\2\\-1\end{bmatrix}, \boldsymbol{z_1} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\1\end{bmatrix}$ and that $\mathfrak{X} = \{\boldsymbol{w_1}, \boldsymbol{w_2}\}$ and $\mathfrak{Y} = \{\boldsymbol{z_1}\}$ are orthonormal bases for W and $W^\perp$, respectively. Let $\boldsymbol{w} = (\boldsymbol{b}\cdot\boldsymbol{w_1})\boldsymbol{w_1} + (\boldsymbol{b}\cdot\boldsymbol{w_2})\boldsymbol{w_2}$ (2 pts)$=\begin{bmatrix}5/6\\7/3\\23/6\end{bmatrix}$ (1 pt), and $\boldsymbol{z} = (\boldsymbol{b}\cdot\boldsymbol{z_1})\boldsymbol{z_1}$ (1 pt) $= \frac{1}{6}\begin{bmatrix}1\\-2\\1\end{bmatrix}$ (1 pt), then $\mathbf{w} \in W$, $\mathbf{z} \in W^\perp$ and $\mathbf{b} = \mathbf{w} + \mathbf{z}$.

6. Let $\boldsymbol{u} = \begin{bmatrix}1\\4\\-1\end{bmatrix}$ and $S = \left\{\frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}$.

a. Check that S is orthonormal. (4 pts)

Let $\boldsymbol{v_1} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\1\end{bmatrix}$, $\boldsymbol{v_2} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}$. Check $\boldsymbol{v_1}\cdot\boldsymbol{v_2} = 0$ (2 pts), $\boldsymbol{v_1}\cdot\boldsymbol{v_1} = 1$ (1 pt), $\boldsymbol{v_2}\cdot\boldsymbol{v_2} = 1$ (1 pt) (details omitted).

b. Find the vector $\boldsymbol{w}$ in the span of S that is closest to $\boldsymbol{u}$. (4 pts)

$\boldsymbol{w} = (\boldsymbol{u} \cdot \boldsymbol{v_1})\boldsymbol{v_1} + (\boldsymbol{u} \cdot \boldsymbol{v_2})\boldsymbol{v_2} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$. (2 pts for correct formula; 2 pts for computation.)

c. Find the distance between $\boldsymbol{w}$ and $\boldsymbol{u}$. (4 pts)

The distance $\|\boldsymbol{w} - \boldsymbol{u}\| = 0$ (2 pts) since $\boldsymbol{w} = \boldsymbol{u}$ (2 pts). ($\boldsymbol{u}$ *is in the span of* S.)