Instructions

1. The statements in Italics are for introducing results and notations that may be used again in this course. You are only required to read and think about them.

2. To receive full credit you must explain how you got your answer.

3. While I encourage collaboration, you must write solutions IN YOUR OWN WORDS. DO NOT SHARE COMPLETE SOLUTIONS before they are due. YOU WILL RECEIVE NO CREDIT if you are found to have copied from whatever source or let others copy your solutions.

4. Homework must be handwritten (electronic handwriting is allowed) for authentication purposes and submitted on Canvas. Please do NOT include any personal information such as your name and netID in your file. Late homework will NOT be accepted. It is your responsibility to MAKE SURE THAT YOUR SUBMISSIONS ARE SUCCESSFUL AND YOUR FILES ARE LEGIBLE AND COMPLETE. It is also your responsibility that whoever reads your work will understand and enjoy it. Up to 4 points out of 40 may be taken off if your solutions are hard to read or poorly presented.

Homework 4 Solution

1. Use the Wronskian to determine whether each of the following sets of functions is linearly dependent. (2 pts each)

a. $\{e^{ax}sin(bx), e^{ax}cos(bx)\}$, where a, b are real numbers.

$$det(\begin{bmatrix} e^{ax}sin(bx) & e^{ax}cos(bx) \\ (e^{ax}sin(bx))' & (e^{ax}cos(bx))' \end{bmatrix})$$
$$= det(\begin{bmatrix} e^{ax}sin(bx) & e^{ax}cos(bx) \\ ae^{ax}sin(bx) + be^{ax}cos(bx) & ae^{ax}cos(bx) - be^{ax}sin(bx) \end{bmatrix})$$
$$= e^{ax}sin(bx)(ae^{ax}cos(bx) - be^{ax}sin(bx)) - e^{ax}cos(bx)(ae^{ax}sin(bx) + be^{ax}cos(bx))$$
$$= -be^{2ax}(sin^2(bx) + cos^2(bx))$$
$= -be^{2ax}$ (Wronskian) (1 pt)

When $b = 0$ the Wronskian is 0 and the functions are linearly dependent (in fact, the first function is 0) (0.5 pt); when $b \neq 0$, the Wronskian is nonzero and the functions are linearly independent. (0.5 pt)

b. $\{1, x, x^2\}$.

$det(\begin{bmatrix} 1 & x & x^2 \\ 1' & x' & (x^2)'' \\ 1'' & x'' & (x^2)'' \end{bmatrix}) = det(\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{bmatrix}) = 2$ (Wronskian) (*The last determinant is very easy to compute by the way*) (1 pt)

Since the Wronskian is nonzero, the functions are linearly independent. (1 pt)

2. Let $\mathfrak{X}$ be the standard basis for $\mathbb{R}^4$ and let

$$\mathfrak{Y} = \{(-1,0,0,0),(0,2,0,0),(0,0,3,0),(0,0,0,4)\}.$$

Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by

$$[T]_{\mathfrak{X}\mathfrak{X}} = \begin{bmatrix} 3 & 0 & 22 & 4 \\ 1 & -1 & \sqrt{3}/3 & -2 \\ -5 & -4 & 19 & -8 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

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a. Compute $det([T]_{\mathfrak{X}\mathfrak{X}})$. (2 pts)

Expand along the last row we get $det([T]_{\mathfrak{X}\mathfrak{X}}) = (-1)^{4+3} 2 det(\begin{bmatrix} 3 & 0 & 4 \\ 1 & -1 & -2 \\ -5 & -4 & -8 \end{bmatrix})$ (1 pt). Expand along the first row in the smaller determinant we get

$$det([T]_{\mathfrak{X}\mathfrak{X}}) = -2[(-1)^{1+1} 3 det(\begin{bmatrix} -1 & -2 \\ -4 & -8 \end{bmatrix}) + (-1)^{1+3} 4 det(\begin{bmatrix} 1 & -1 \\ -5 & -4 \end{bmatrix})] = 72$$

(1 pt).

b. Compute $[id_{\mathbb{R}^4}]_{\mathfrak{X}\mathfrak{Y}}, [T]_{\mathfrak{X}\mathfrak{Y}}$ and $[T]_{\mathfrak{Y}\mathfrak{Y}}$. (5 pts)

$[id_{\mathbb{R}^4}]_{\mathfrak{X}\mathfrak{Y}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ (details omitted) (1 pt)

$T((-1,0,0,0)) = T(-1 \cdot (1,0,0,0)) = -1 \cdot T((1,0,0,0))$
$= -1 \cdot [3 \cdot (1,0,0,0) + 1 \cdot (0,1,0,0) + (-5) \cdot (0,0,1,0) + 0 \cdot (0,0,0,1)]$
$= (-3) \cdot (1,0,0,0) + (-1) \cdot (0,1,0,0) + 5 \cdot (0,0,1,0) + 0 \cdot (0,0,0,1)$;
$T((0,2,0,0)) = T(2 \cdot (0,1,0,0)) = 2 \cdot T((0,1,0,0))$
$= 2 \cdot [0 \cdot (1,0,0,0) + (-1) \cdot (0,1,0,0) + (-4) \cdot (0,0,1,0) + 0 \cdot (0,0,0,1)]$
$= 0 \cdot (1,0,0,0) + (-2) \cdot (0,1,0,0) + (-8) \cdot (0,0,1,0) + 0 \cdot (0,0,0,1)$;
$T((0,0,3,0)) = T(3 \cdot (0,0,1,0)) = 3 \cdot T((0,0,1,0))$
$= 3 \cdot [22 \cdot (1,0,0,0) + \sqrt{3}/3 \cdot (0,1,0,0) + 19 \cdot (0,0,1,0) + 2 \cdot (0,0,0,1)$
$= 66 \cdot (1,0,0,0) + \sqrt{3} \cdot (0,1,0,0) + 57 \cdot (0,0,1,0) + 6 \cdot (0,0,0,1)$;
$T((0,0,0,4)) = T(4 \cdot (0,0,0,1)) = 4 \cdot T((0,0,0,1))$
$= 4 \cdot [4 \cdot (1,0,0,0) + (-2) \cdot (0,1,0,0) + (-8) \cdot (0,0,1,0) + 0 \cdot (0,0,0,1)]$
$= 16 \cdot (1,0,0,0) + (-8) \cdot (0,1,0,0) + (-32) \cdot (0,0,1,0) + 0 \cdot (0,0,0,1)$. (1 pt)
(Alternatively, $[T]_{\mathfrak{X}\mathfrak{Y}} = [T]_{\mathfrak{X}\mathfrak{X}}[id_{\mathbb{R}^4}]_{\mathfrak{X}\mathfrak{Y}}$.)

Therefore, $[T]_{\mathfrak{X}\mathfrak{Y}} = \begin{bmatrix} -3 & 0 & 66 & 16 \\ -1 & -2 & \sqrt{3} & -8 \\ 5 & -8 & 57 & -32 \\ 0 & 0 & 6 & 0 \end{bmatrix}$ (1 pt).

$T((-1,0,0,0)) = (-3) \cdot (1,0,0,0) + (-1) \cdot (0,1,0,0) + 5 \cdot (0,0,1,0) + 0 \cdot (0,0,0,1) = 3 \cdot (-1,0,0,0) + (-1/2) \cdot (0,2,0,0) + 5/3 \cdot (0,0,3,0) + 0 \cdot (0,0,0,4)$;
$T((0,2,0,0)) = 0 \cdot (1,0,0,0) + (-2) \cdot (0,1,0,0) + (-8) \cdot (0,0,1,0) + 0 \cdot (0,0,0,1) = 0 \cdot (-1,0,0,0) + (-1) \cdot (0,2,0,0) + (-8/3) \cdot (0,0,3,0) + 0 \cdot (0,0,0,4)$;
$T((0,0,3,0)) = 66 \cdot (1,0,0,0) + \sqrt{3} \cdot (0,1,0,0) + 57 \cdot (0,0,1,0) + 6 \cdot (0,0,0,1) = -66 \cdot (-1,0,0,0) + \sqrt{3}/2 \cdot (0,2,0,0) + 19 \cdot (0,0,3,0) + 3/2 \cdot (0,0,0,4)$;

$T((0,0,0,4)) = 16\cdot(1,0,0,0)+(-8)\cdot(0,1,0,0)+(-32)\cdot(0,0,1,0)+0\cdot(0,0,0,1) = -16\cdot(-1,0,0,0)+(-4)\cdot(0,2,0,0)+(-32/3)\cdot(0,0,3,0)+0\cdot(0,0,0,4)$. (1 pt)

Alternatively, $[T]_{\mathfrak{Y}\mathfrak{Y}} = [id_{\mathbb{R}^4}]_{\mathfrak{Y}\mathfrak{X}}[T]_{\mathfrak{X}\mathfrak{Y}}$.

Therefore, $[T]_{\mathfrak{Y}\mathfrak{Y}} = \begin{bmatrix} 3 & 0 & -66 & -16 \\ -1/2 & -1 & \sqrt{3}/2 & -4 \\ 5/3 & -8/3 & 19 & 32/3 \\ 0 & 0 & 3/2 & 0 \end{bmatrix}$ (1 pt).

c. Compute $det([id_{\mathbb{R}^4}]_{\mathfrak{X}\mathfrak{Y}}), det([T]_{\mathfrak{X}\mathfrak{Y}})$ and $det([T]_{\mathfrak{Y}\mathfrak{Y}})$ by definition. Verify that $det([T]_{\mathfrak{Y}\mathfrak{Y}}) = det([T]_{\mathfrak{X}\mathfrak{X}})$, $det([T]_{\mathfrak{X}\mathfrak{Y}}) = det([T]_{\mathfrak{X}\mathfrak{X}})det([id_{\mathbb{R}^4}]_{\mathfrak{X}\mathfrak{Y}})$. (7 pts)

$det([id_{\mathbb{R}^4}]) = -24, det([T]_{\mathfrak{X}\mathfrak{Y}}) = -1728, det([T])_{\mathfrak{Y}\mathfrak{Y}} = 72$ (details omitted) (2 pts each). Therefore, $det([T]_{\mathfrak{Y}\mathfrak{Y}}) = det([T]_{\mathfrak{X}\mathfrak{X}})$, $det([T]_{\mathfrak{X}\mathfrak{X}})det([id_{\mathbb{R}^4}]_{\mathfrak{X}\mathfrak{Y}}) = 72 \times (-24) = -1728 = det([T]_{\mathfrak{X}\mathfrak{Y}})$. (1 pt)

d. Compute $det([id_{\mathbb{R}^4}]_{\mathfrak{Y}\mathfrak{X}})$ without computing $[id_{\mathbb{R}^4}]_{\mathfrak{Y}\mathfrak{X}}$. (4 pts)

$$[id_{\mathbb{R}^4}]_{\mathfrak{Y}\mathfrak{X}}[id_{\mathbb{R}^4}]_{\mathfrak{X}\mathfrak{Y}} = [id_{\mathbb{R}^4} \circ id_{\mathbb{R}^4}]_{\mathfrak{Y}\mathfrak{Y}} = [id_{\mathbb{R}^4}]_{\mathfrak{Y}\mathfrak{Y}},$$

so

$$det([id_{\mathbb{R}^4}]_{\mathfrak{Y}\mathfrak{X}}[id_{\mathbb{R}^4}]_{\mathfrak{X}\mathfrak{Y}}) = det([id_{\mathbb{R}^4}]_{\mathfrak{Y}\mathfrak{X}})det([id_{\mathbb{R}^4}]_{\mathfrak{X}\mathfrak{Y}}) = det([id_{\mathbb{R}^4}]_{\mathfrak{Y}\mathfrak{Y}}) = det(I_4) = 1,$$

and

$$det([id_{\mathbb{R}^4}]_{\mathfrak{Y}\mathfrak{X}}) = 1/det([id_{\mathbb{R}^4}]_{\mathfrak{X}\mathfrak{Y}}) = -1/24.$$

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Each equality in the middle line is worth 1 pt.

3. Recall that $\mathcal{P}_2$ is the vector space of all polynomials of the form $a_0 + a_1x + a_2x^2$, where a_0, a_1, a_2 are real numbers. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation given by differentiation. Compute all eigenvalues and eigenspaces of T. Is T diagonalizable? (6 pts)

Let $\mathfrak{X} = \{1, x, x^2\}$, then $[T]_{\mathfrak{X}\mathfrak{X}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ (details omitted) (1 pt).

$det([T]_{\mathfrak{X}\mathfrak{X}} - \lambda I) = \lambda^3$. (1 pt) So the only eigenvalue is 0 (*with multiplicity* 3). (1 pt)

$[T]_{\mathfrak{X}\mathfrak{X}} - 0I = [T]_{\mathfrak{X}\mathfrak{X}}$, so the general solution to $([T]_{\mathfrak{X}\mathfrak{X}} - 0I)\mathbf{v} = \mathbf{0}$ is $\begin{cases} v_2 = 0 \\ v_3 = 0 \\ v_1 \in \mathbb{R} \end{cases}$ (1 pt), and the only eigenspace $E_0 = \{v_1 \cdot 1 + 0 \cdot x + 0 \cdot x^2, v_1 \in \mathbb{R}\}$ (1 pt). E_0 is a 1−dimensional subspace of $\mathcal{P}_2$ with basis $\{1\}$. Since the total dimension of eigenspaces (1) is smaller than the dimension of $\mathcal{P}_2$ (3), T is not diagonalizable. (1 pt)

4. a. For the 2×2 diagonal matrix $X = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}$, compute $a_0 I_2 + a_1 X + ... a_n X^n$, where $x_1, x_2, a_0, ..., a_n$ are real numbers and I_2 is the 2×2 identity matrix. (2 pts)

$= \begin{bmatrix} a_0 + a_1 x_1 + ... + a_n x_1^n & 0 \\ 0 & a_0 + a_1 x_2 + ... + a_n x_2^n \end{bmatrix}$ (details ommitted; it's not hard to see the pattern when you compute small powers of X and take linear combinations) (2 pts).

b. Compute $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$. (4 pts)

Computing this directly is hard, but part a inspires us to diagonalize the matrix. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then A has eigenvalues $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$; let $a = \frac{1+\sqrt{5}}{2}$, then the corresponding eigenvectors are $\begin{bmatrix} a \\ 1 \end{bmatrix}, \begin{bmatrix} -1/a \\ 1 \end{bmatrix}$, respectively (details omitted) (1 pt). Let $\mathfrak{X} = \{(a,1), (-1/a, 1)\}$, $\mathfrak{Y} = \{(1,0),(0,1)\}$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $[T]_{\mathfrak{Y}\mathfrak{Y}} = A$, then $[T]_{\mathfrak{X}\mathfrak{X}} = \begin{bmatrix} a & 0 \\ 0 & -1/a \end{bmatrix}$ (which we denote by D) and $[T]_{\mathfrak{Y}\mathfrak{Y}} = [id_{\mathbb{R}^2}]_{\mathfrak{Y}\mathfrak{X}} [T]_{\mathfrak{X}\mathfrak{X}} [id_{\mathbb{R}^2}]_{\mathfrak{X}\mathfrak{Y}}$. Let $Q = [id_{\mathbb{R}^2}]_{\mathfrak{Y}\mathfrak{X}}$, then $A = QDQ^{-1}$. $A^n = (QDQ^{-1})^n = QD^nQ^{-1}$ (the last equality holds because matrix multiplication is associative and $Q^{-1}Q = I$) (1 pt). Now $Q = \begin{bmatrix} a & -1/a \\ 1 & 1 \end{bmatrix}$(1 pt), and $Q^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -1/a \\ -1 & a \end{bmatrix}$ (details omitted), so

$A^n = \begin{bmatrix} a & -1/a \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a^n & 0 \\ 0 & (-1/a)^n \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -1/a \\ -1 & a \end{bmatrix}$

$= \frac{1}{\sqrt{5}} \begin{bmatrix} a^{n+1} - (1/a)^{n+1} & -a^n - (-1/a)^n \\ a^n - (-1/a)^n & -a^{n-1} - (-1/a)^{n-1} \end{bmatrix}$ (1 pt).

c. The Fibonacci sequence is defined as $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for each integer $n \geq 2$. Note that $F_n = F_{n-1} + F_{n-2}$ is equivalent to $\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$. Find a (non-recursive) formula for F_n. (2 pts) *To check your answer,* $\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$ *should be the golden ratio* $\frac{1+\sqrt{5}}{2}$.

$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} a^n - (1/a)^n & -a^{n-1} - (-1/a)^{n-1} \\ a^{n-1} - (-1/a)^{n-1} & -a^{n-2} - (-1/a)^{n-2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (1 pt) $= \frac{1}{\sqrt{5}} \begin{bmatrix} a^n - (1/a)^n \\ a^{n-1} - (-1/a)^{n-1} \end{bmatrix}$. Therefore, $F_n = \frac{1}{\sqrt{5}}[a^n - (1/a)^n]$ (1 pt).

d. Find a "square root" of $B = \begin{bmatrix} -8 & 6 \\ -18 & 13 \end{bmatrix}$, i.e., find a 2×2 matrix A such that $A^2 = B$. (You are not allowed to start with an answer and then check that it works; you must explain how you found A.) (4 pts)

$B = QDQ^{-1}$, where $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ (details omitted) (eigenvalues and eigenvectors 1 pt; diagonalization 1 pt). Let $A = Q \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} Q^{-1}$ (can replace 1 with -1 and/or 2 with -2) (1 pt) $= \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix}$ (1 pt), then $A^2 = QDQ^{-1} = B$.