Instructions

1. The statements in Italics are for introducing results and notations that may be used again in this course. You are only required to read and think about them.

2. To receive full credit you must explain how you got your answer.

3. While I encourage collaboration, you must write solutions IN YOUR OWN WORDS. DO NOT SHARE COMPLETE SOLUTIONS before they are due. YOU WILL RECEIVE NO CREDIT if you are found to have copied from whatever source or let others copy your solutions.

4. Homework must be handwritten (electronic handwriting is allowed) for authentication purposes and submitted on Canvas. Please do NOT include any personal information such as your name and netID in your file. Late homework will NOT be accepted. It is your responsibility to MAKE SURE THAT YOUR SUBMISSIONS ARE SUCCESSFUL AND YOUR FILES ARE LEGIBLE AND COMPLETE. It is also your responsibility that whoever reads your work will understand and enjoy it. Up to 4 points out of 40 may be taken off if your solutions are hard to read or poorly presented.

Homework 4

1. Use the Wronskian to determine whether each of the following sets of functions is linearly dependent. (2 pts each)

a. $\{e^{ax}sin(bx), e^{ax}cos(bx)\}$, where a, b are real numbers.

b. $\{1, x, x^2\}$.

2. Let $\mathfrak{X}$ be the standard basis for $\mathbb{R}^4$ and let

$$\mathfrak{Y} = \{(-1, 0, 0, 0), (0, 2, 0, 0), (0, 0, 3, 0), (0, 0, 0, 4)\}.$$

Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by

$$[T]_{\mathfrak{X}\mathfrak{X}} = \begin{bmatrix} 3 & 0 & 22 & 4 \\ 1 & -1 & \sqrt{3}/3 & -2 \\ -5 & -4 & 19 & -8 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

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a. Compute $det([T]_{\mathfrak{X}\mathfrak{X}})$. (2 pts)

b. Compute $[id_{\mathbb{R}^4}]_{\mathfrak{X}\mathfrak{Y}}, [T]_{\mathfrak{X}\mathfrak{Y}}$ and $[T]_{\mathfrak{Y}\mathfrak{Y}}$. (5 pts)

c. Compute $det([id_{\mathbb{R}^4}]_{\mathfrak{X}\mathfrak{Y}}), det([T]_{\mathfrak{X}\mathfrak{Y}})$ and $det([T]_{\mathfrak{Y}\mathfrak{Y}})$ by definition. Verify that $det([T]_{\mathfrak{Y}\mathfrak{Y}}) = det([T]_{\mathfrak{X}\mathfrak{X}})$, $det([T]_{\mathfrak{X}\mathfrak{Y}}) = det([T]_{\mathfrak{X}\mathfrak{X}})det([id_{\mathbb{R}^4}]_{\mathfrak{X}\mathfrak{Y}})$. (7 pts)

d. Compute $det([id_{\mathbb{R}^4}]_{\mathfrak{Y}\mathfrak{X}})$ without computing $[id_{\mathbb{R}^4}]_{\mathfrak{Y}\mathfrak{X}}$. (4 pts)

3. Recall that $\mathcal{P}_2$ is the vector space of all polynomials of the form $a_0 + a_1x + a_2x^2$, where a_0, a_1, a_2 are real numbers. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation given by differentiation. Compute all eigenvalues and eigenspaces of T. Is T diagonalizable? (6 pts)

4. a. For the 2×2 diagonal matrix $X = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}$, compute $a_0 I_2 + a_1 X + ... a_n X^n$, where $x_1, x_2, a_0, ..., a_n$ are real numbers and I_2 is the 2×2 identity matrix. (2 pts)

b. Compute $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$. (4 pts)

c. The Fibonacci sequence is defined as $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for each integer $n \geq 2$. Note that $F_n = F_{n-1} + F_{n-2}$ is equivalent to $\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$. Find a (non-recursive) formula for F_n. (2 pts) *To check your answer,* $\lim_{n \to \infty} \frac{F_{n+1}}{F_n}$ *should be the golden ratio* $\frac{1+\sqrt{5}}{2}$.

d. Find a "square root" of $B = \begin{bmatrix} -8 & 6 \\ -18 & 13 \end{bmatrix}$, i.e., find a 2×2 matrix A such that $A^2 = B$. (You are not allowed to start with an answer and then check that it works; you must explain how you found A.) (4 pts)