Instructions

1. The statements in Italics are for introducing results and notations that may be used again in this course. You are only required to read and think about them.

2. To receive full credit you must explain how you got your answer.

3. While I encourage collaboration, you must write solutions IN YOUR OWN WORDS. DO NOT SHARE COMPLETE SOLUTIONS before they are due. YOU WILL RECEIVE NO CREDIT if you are found to have copied from whatever source or let others copy your solutions.

4. Homework must be handwritten (electronic handwriting is allowed) for authentication purposes and submitted on Canvas. Please do NOT include any personal information such as your name and netID in your file. Late homework will NOT be accepted. It is your responsibility to MAKE SURE THAT YOUR SUBMISSIONS ARE SUCCESSFUL AND YOUR FILES ARE LEGIBLE AND COMPLETE. It is also your responsibility that whoever reads your work will understand and enjoy it. Up to 4 points out of 40 may be taken off if your solutions are hard to read or poorly presented.

Homework 3 Solution

1. Using Gaussian Elimination, either find the general or unique solution of the system, or explain why it is **inconsistent** (i.e., has no solution) (2 pts each):

a. $\begin{cases} -2x_2 + x_3 + 2x_4 = 0 \\ 2x_1 + x_3 + 4x_4 = 1 \end{cases}$

We do Gaussian elimination on the matrix $\left[\begin{array}{cccc|c} 0 & -2 & 1 & 2 & 0 \\ 2 & 0 & 1 & 4 & 1 \end{array}\right]$. Switch the two rows to get $\left[\begin{array}{cccc|c} 2 & 0 & 1 & 4 & 1 \\ 0 & -2 & 1 & 2 & 0 \end{array}\right]$; scale the two rows by $1/2$ and $-1/2$, respectively to get $\left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 2 & 1/2 \\ 0 & 1 & -1/2 & -1 & 0 \end{array}\right]$, which is the reduced row echelon form. From it we see that x_3 and x_4 are free variables, $x_1 + \frac{1}{2}x_3 + 2x_4 = \frac{1}{2}$, and $x_2 - \frac{1}{2}x_3 - x_4 = 0$. Therefore, the general solution is $\begin{cases} x_1 = \frac{1}{2} - \frac{1}{2}x_3 - 2x_4 \\ x_2 = \frac{1}{2}x_3 + x_4 \\ x_3, x_4 \in \mathbb{R} \end{cases}$.

b. $\begin{cases} -2x_2 + x_3 + 2x_4 = 0 \\ 2x_1 + x_3 + 4x_4 = 1 \\ x_1 + x_4 = 2 \\ 2x_1 - x_2 + x_4 = -4 \end{cases}$

Since the first two equations are the same as in part a, we can start with the matrix $\left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 2 & 1/2 \\ 0 & 1 & -1/2 & -1 & 0 \\ 1 & 0 & 0 & 1 & 2 \\ 2 & -1 & 0 & 1 & -4 \end{array}\right]$. Subtract row 1 from row 3 and 2 times row 1 from row 4 to get $\left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 2 & 1/2 \\ 0 & 1 & -1/2 & -1 & 0 \\ 0 & 0 & -1/2 & -1 & 3/2 \\ 0 & -1 & -1 & -3 & -5 \end{array}\right]$; add row 2 to row 4 to get $\left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 2 & 1/2 \\ 0 & 1 & -1/2 & -1 & 0 \\ 0 & 0 & -1/2 & -1 & 3/2 \\ 0 & 0 & -3/2 & -4 & -5 \end{array}\right]$; scale row 3 by -2 to get $\left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 2 & 1/2 \\ 0 & 1 & -1/2 & -1 & 0 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & -3/2 & -4 & -5 \end{array}\right]$; add 3/2 times row 3 to row 4 to get $\left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 2 & 1/2 \\ 0 & 1 & -1/2 & -1 & 0 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & -1 & -19/2 \end{array}\right]$; scale row 4 by -1 to get $\left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 2 & 1/2 \\ 0 & 1 & -1/2 & -1 & 0 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 19/2 \end{array}\right]$; add row 4 to row 2 and subtract 2 times row 4 to row 1 and row 3 to get $\left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 0 & -37/2 \\ 0 & 1 & -1/2 & 0 & 19/2 \\ 0 & 0 & 1 & 0 & -22 \\ 0 & 0 & 0 & 1 & 19/2 \end{array}\right]$; add and subtract 1/2 times row 3 to row 2 and row 1, respectively to get $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -15/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & -22 \\ 0 & 0 & 0 & 1 & 19/2 \end{array}\right]$ (reduced row echelon form). Therefore, the unique solution is $\begin{cases} x_1 = -15/2 \\ x_2 = -3/2 \\ x_3 = -22 \\ x_4 = 19/2 \end{cases}$.

c. $\begin{cases} -2x_2 + x_3 + 2x_4 = 0 \\ 2x_1 + x_3 + 4x_4 = 1 \\ x_1 + x_4 = 2 \\ 2x_1 - x_2 + x_4 = -4 \\ 2x_1 - x_2 + x_3 + 3x_4 = -7 \end{cases}$

Since the first four equations are the same as in part b, we can start with the matrix $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -15/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & -22 \\ 0 & 0 & 0 & 1 & 19/2 \\ 2 & -1 & 1 & 3 & -7 \end{array}\right]$. Subtract 2 times row 1 from row 5 to get

$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -15/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & -22 \\ 0 & 0 & 0 & 1 & 19/2 \\ 0 & -1 & 1 & 3 & 8 \end{array}\right]$; add row 2 to row 5 to get $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -15/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & -22 \\ 0 & 0 & 0 & 1 & 19/2 \\ 0 & 0 & 1 & 3 & 13/2 \end{array}\right]$;

subtract row 4 from row 5 to get $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -15/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & -22 \\ 0 & 0 & 0 & 1 & 19/2 \\ 0 & 0 & 0 & 3 & 57/2 \end{array}\right]$; subtract 3 times row 4 from row 5 to get $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -15/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & -22 \\ 0 & 0 & 0 & 1 & 19/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ (reduced row echelon form). Therefore, the unique solution is $\begin{cases} x_1 = -15/2 \\ x_2 = -3/2 \\ x_3 = -22 \\ x_4 = 19/2 \end{cases}$.

d. $\begin{cases} -2x_2 + x_3 + 2x_4 = 0 \\ 2x_1 + x_3 + 4x_4 = 1 \\ x_1 + x_4 = 2 \\ 2x_1 - x_2 + x_4 = -4 \\ 2x_1 - x_2 + x_3 + 3x_4 = -7 \\ -3x_1 - 2x_2 + x_3 - x_4 = -5 \end{cases}$

Since the first five equations are the same as in part c, we can start with the matrix $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -15/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & -22 \\ 0 & 0 & 0 & 1 & 19/2 \\ 0 & 0 & 0 & 0 & 0 \\ -3 & -2 & 1 & -1 & -5 \end{array}\right]$. Add 3 times row 1 to row 6 to get $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -15/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & -22 \\ 0 & 0 & 0 & 1 & 19/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 & -55/2 \end{array}\right]$; add 2 times row 2 to row 5 to get $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -15/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & -22 \\ 0 & 0 & 0 & 1 & 19/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -55/2 \end{array}\right]$; subtract row 3 from row 6 to get $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -15/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & -22 \\ 0 & 0 & 0 & 1 & 19/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -11/2 \end{array}\right]$;

add row 5 to row 6 to get $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -15/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & -22 \\ 0 & 0 & 0 & 1 & 19/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right]$. Since $0 \neq 4$, this system of equations is inconsistent.

2. $A = \begin{bmatrix} -1 & -2 & -3 & -4 \\ 0 & 2 & 3 & 0 \\ 0 & 2 & 0 & -3 \\ 1 & -2 & 0 & 7 \end{bmatrix}$. Use the matrix inversion algorithm to find the inverse of A or show that A is not invertible (4 pts).

Start with $\left[\begin{array}{cccc|cccc} -1 & -2 & -3 & -4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -3 & 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & 7 & 0 & 0 & 0 & 1 \end{array}\right]$ (1 pt). Add row 1 to row 4 to get $\left[\begin{array}{cccc|cccc} -1 & -2 & -3 & -4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -3 & 0 & 0 & 1 & 0 \\ 0 & -4 & -3 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$; subtract row 2 from row 3 and add 2 times row 2 to row 4 to get $\left[\begin{array}{cccc|cccc} -1 & -2 & -3 & -4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 & 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 3 & 1 & 2 & 0 & 1 \end{array}\right]$ (2 pts). From this we see that the rows in A are linearly dependent. Therefore, A is not invertible (1 pt).

3. Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation given by the matrix

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 & 2 & 4 & 5 & -3 \\ 2 & 5 & 10 & 11 & -7 \\ 1 & 1 & 2 & 4 & -2 \end{bmatrix}$$

under standard bases (i.e., this is the matrix $[T]_{\mathfrak{X}\mathfrak{Y}}$, where

$$\mathfrak{X} = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$$

and

$$\mathfrak{Y} = \{(1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1)\}$$

are the standard basis for $\mathbb{R}^4$ and $\mathbb{R}^5$, respectively).

a. Find a basis for $K(T)$ and $R(T)$, respectively (3 pts each).

To compute $K(T)$ we need to solve the equation $[T]_{\mathfrak{X}\mathfrak{Y}}\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$, $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Performing elementary row operations on $[T]_{\mathfrak{X}\mathfrak{Y}}$ doesn't change the set of solutions to the equations. Therefore, we can use Gaussian elimination (details skipped) to obtain the reduced row echelon form of $[T]_{\mathfrak{X}\mathfrak{Y}}$: $\begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. (1 pt) From this we see that x_3 and x_4 are free variables, and $x_1 + 3x_4 = 0$, $x_2 + 2x_3 + x_4 = 0$, $x_5 = 0$. Therefore, $K(T) = \left\{ \begin{bmatrix} -3x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix}, x_3, x_4 \in \mathbb{R} \right\}$ (1 pt) with basis $\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ (1 pt).

$R(T)$ is the column span of $[T]_{\mathfrak{X}\mathfrak{Y}}$. Method 1: Since elementary column operations preserve the column span, we can use them (details skipped) to get the reduced column echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 \end{bmatrix}$ (you can also get this by doing the usual Gaussian elimination on the transpose of $[T]_{\mathfrak{X}\mathfrak{Y}}$ and transpose back) (2 pts). From this we see that a basis for $R(T)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ (1 pt). Method 2: The column correspondence property implies that $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -7 \\ -2 \end{bmatrix} \right\}$ is a basis for the column span or $R(T)$ (3 pts).

b. Find the reduced row echelon form of the matrix (1 pt).

It is $\begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

c. What is the rank of the matrix (1 pt)?

Since there are 3 pivot 1's in the reduced row/column echelon form (or since $R(T)$ has dimension 3), the rank is 3.

4. a. Use Gaussian elimination to show that $\mathfrak{X} = \{(0,2,4),(1,2,3),(5,2,0)\}$ and $\mathfrak{Y} = \{(1,2,3),(0,1,4),(5,6,0)\}$ are bases for $\mathbb{R}^3$ (3 pts each).

We perform Gaussian elimination on $\begin{bmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \\ 5 & 2 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ (details (2 pts each) skipped) to get their reduced row echelon forms. Both forms are I_3 (1 pt each). Therefore, the vectors in $\mathfrak{X}$ and in $\mathfrak{Y}$ are linearly independent and $\mathfrak{X}$, $\mathfrak{Y}$ each span $\mathbb{R}^3$.

b. Let $\mathfrak{Z} = \{(1,0,0),(0,1,0),(0,0,1)\}$. Use the matrix inversion algorithm to compute $[id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{Z}}$ and $[id_{\mathbb{R}^3}]_{\mathfrak{Y}\mathfrak{Z}}$ (3 pts each).

Since $\begin{cases} id_{\mathbb{R}^3}((0,2,4)) = (0,2,4) = 0\cdot(1,0,0) + 2\cdot(0,1,0) + 4\cdot(0,0,1) \\ id_{\mathbb{R}^3}((1,2,3)) = (0,2,4) = 1\cdot(1,0,0) + 2\cdot(0,1,0) + 3\cdot(0,0,1) \\ id_{\mathbb{R}^3}((5,2,0)) = (5,2,0) = 5\cdot(1,0,0) + 2\cdot(0,1,0) + 0\cdot(0,0,1) \end{cases}$,

$[id_{\mathbb{R}^3}]_{\mathfrak{Z}\mathfrak{X}} = \begin{bmatrix} 0 & 1 & 5 \\ 2 & 2 & 2 \\ 4 & 3 & 0 \end{bmatrix}$ (1 pt). Similarly, $[id_{\mathbb{R}^3}]_{\mathfrak{Z}\mathfrak{Y}} = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$ (1 pt). Since $[id_{\mathbb{R}^3}]_{\mathfrak{Z}\mathfrak{X}}[id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{Z}} = [id_{\mathbb{R}^3} \circ id_{\mathbb{R}^3}]_{\mathfrak{Z}\mathfrak{Z}} = [id_{\mathbb{R}^3}]_{\mathfrak{Z}\mathfrak{Z}} = I_3 = [id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{X}} = [id_{\mathbb{R}^3} \circ id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{X}} = [id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{Z}}[id_{\mathbb{R}^3}]_{\mathfrak{Z}\mathfrak{X}}$, $[id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{Z}} = ([id_{\mathbb{R}^3}]_{\mathfrak{Z}\mathfrak{X}})^{-1}$ (1 pt). Similarly, $[id_{\mathbb{R}^3}]_{\mathfrak{Y}\mathfrak{Z}} = ([id_{\mathbb{R}^3}]_{\mathfrak{Z}\mathfrak{Y}})^{-1}$ (1 pt). Start with $\left[\begin{array}{ccc|ccc} 0 & 1 & 5 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$; switch the first two rows to get $\left[\begin{array}{ccc|ccc} 2 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 0 \\ 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$; subtract 2 times row 1 from row 3 to get $\left[\begin{array}{ccc|ccc} 2 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 0 \\ 0 & -1 & -4 & 0 & -2 & 1 \end{array}\right]$; scale row 1 by 1/2 and add row 2 to row 3 to get $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1/2 & 0 \\ 0 & 1 & 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$; subtract row 3 from row 1 and 5 times row 3 from row 2 to get $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 5/2 & -1 \\ 0 & 1 & 0 & -4 & 10 & -5 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$; subtract row 2 from row 1 to get $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -15/2 & 4 \\ 0 & 1 & 0 & -4 & 10 & -5 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$. Therefore, $[id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{Z}} = \begin{bmatrix} 3 & -15/2 & 4 \\ -4 & 10 & -5 \\ 1 & -2 & 1 \end{bmatrix}$ (1 pt). Similarly, $[id_{\mathbb{R}^3}]_{\mathfrak{Y}\mathfrak{Z}} = \begin{bmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{bmatrix}$ (1 pt).

c. Compute $[id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{Y}}$ (2 pts).

$$[id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{Y}} = id_{\mathbb{R}^3} \circ id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{Y}} = [id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{Z}}[id_{\mathbb{R}^3}]_{\mathfrak{Z}\mathfrak{Y}} \text{ (1 pt)}= \begin{bmatrix} 0 & 17/2 & -30 \\ 1 & -10 & 40 \\ 0 & 2 & -7 \end{bmatrix} \text{ (1 pt).}$$

d. Write each of $[id_{\mathbb{R}^3}]_{\mathfrak{Z}\mathfrak{X}}$ and $[id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{Z}}$ as a product of elementary matrices (3 pts each).

Collecting the elementary row operations in part b and write down their matrices in order, we have $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} [id_{\mathbb{R}^3}]_{\mathfrak{Z}\mathfrak{X}} = I_3$ (2 pts). Therefore, $[id_{\mathbb{R}^3}]_{\mathfrak{X}\mathfrak{Z}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (2 pts),
$[id_{\mathbb{R}^3}]_{\mathfrak{Z}\mathfrak{X}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (2 pts).