Instructions

1. The statements in Italics are for introducing results and notations that may be used again in this course. You are only required to read and think about them.

2. To receive full credit you must explain how you got your answer.

3. While I encourage collaboration, you must write solutions IN YOUR OWN WORDS. DO NOT SHARE COMPLETE SOLUTIONS before they are due. YOU WILL RECEIVE NO CREDIT if you are found to have copied from whatever source or let others copy your solutions.

4. Homework must be handwritten (electronic handwriting is allowed) for authentication purposes and submitted on Canvas. Please do NOT include any personal information such as your name and netID in your file. Late homework will NOT be accepted. It is your responsibility to MAKE SURE THAT YOUR SUBMISSIONS ARE SUCCESSFUL AND YOUR FILES ARE LEGIBLE AND COMPLETE. It is also your responsibility that whoever reads your work will understand and enjoy it. Up to 4 points out of 40 may be taken off if your solutions are hard to read or poorly presented.

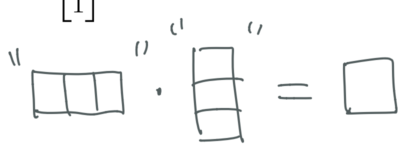
Homework 2 Solution

1. Compute the following products of partitioned matrices using block multiplication:

a. $\begin{bmatrix} 1 & 3 & 1 \end{bmatrix} \left[\begin{array}{c|c} 1 & 2 \\ -1 & 1 \\ 0 & 1 \end{array}\right]$. (2 pts)

$\begin{bmatrix} 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = [-2]$ (1 pt), $\begin{bmatrix} 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = [6]$ (1 pt) so $[-2|6]$.

b. $\left[\begin{array}{c|c|c} 1 & -1 & 0 \\ 0 & 1 & 2 \end{array}\right] \left[\begin{array}{c} 1 \\ \hline 3 \\ \hline 2 \end{array}\right]$. (2 pts)



$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1] + \begin{bmatrix} -1 \\ 1 \end{bmatrix} [3] + \begin{bmatrix} 0 \\ 2 \end{bmatrix} [2]$ (1 pt) $= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$ (1 pt).

c. $\left[\begin{array}{cc} 3 & 0 \\ 0 & 3 \\ \hline 2 & 0 \\ 0 & 2 \end{array}\right] \left[\begin{array}{c|c} 1 & 2 \\ 3 & 4 \end{array}\right]$. (2 pts)

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$ (1 pt), $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$ (1 pt), so $\left[\begin{array}{c|c} 3 & 6 \\ 9 & 12 \\ \hline 2 & 4 \\ 6 & 8 \end{array}\right]$.

2. Compute $\begin{bmatrix} I_n & O \\ B & C \end{bmatrix} \begin{bmatrix} A & B^{-1} \\ O & I_n \end{bmatrix}$, where A, B, C are $n \times n$ matrices, O is the $n \times n$ **zero matrix** (i.e. a matrix whose entries are all 0), and I_n is the $n \times n$ identity matrix. (4 pts)

$$= \begin{bmatrix} I_nA + OO & I_nB^{-1} + OI_n \\ BA + CO & BB^{-1} + CI_n \end{bmatrix} = \begin{bmatrix} A & B^{-1} \\ BA & I_n + C \end{bmatrix} \text{ (each entry 1 pt).}$$

3. Let $S : \mathcal{P}_1 \to \mathcal{P}_2$ be the linear transformation given by $S(p(x)) = \int_0^x p(t)dt$ for all $p(x)$ in $\mathcal{P}_1$. Let $\mathfrak{X} = \{1, x\}$ and $\mathfrak{Y} = \{1, x, x^2\}$. Find a basis for $K(S)$ and a basis for $R(S)$. (6 pts)

"0" vector space

when $\quad$ 0 ? $\quad$ $a_0 = a_1 = 0.$

For a_0, a_1 in $\mathbb{R}$, $S(a_0 + a_1 x) = \int_0^x (a_0 + a_1 t)dt = a_0 x + \frac{1}{2} a_1 x^2$. (2 pts) $\quad$ $=0$

The above expression is equal to 0 only when $a_0 = a_1 = 0$. Therefore, $K(S) = \{0\}$ and has basis the empty set. (Reasoning 1 pt, conclusion 1 pt)

The above expression also tells us that $R(S)$ consists of the polynomials on $\mathcal{P}_2$ without the constant term. So a basis for $R(S)$ is $\{x, x^2\}$. (Reasoning 1 pt, conclusion 1 pt)

$T(a_0 + a_1 x + a_2 x^2) = \underset{0}{\underset{\|}{a_1}} + 2\underset{0}{\underset{\|}{a_2}} x = 0$

4. *Verify that $\mathbb{C}$, together with the ususal addition and scalar multiplication, is a vector space over $\mathbb{R}$.*

a. Show that $\mathfrak{X} = \{1, i\}$ and $\mathfrak{Y} = \{1, i+1\}$ are bases for $\mathbb{C}$. (8 pts)

$a + bi = x \cdot 1 + y \cdot (i+1)$
$= (x+y) + yi$
$a = x+y \quad b = y$
$x = a - b$

We need to show $\mathfrak{X}, \mathfrak{Y}$ each has span $\mathbb{C}$ (1) and is linearly independent (2).

Since each complex number $a + bi = a \cdot 1 + b \cdot i = (a - b) \cdot 1 + b \cdot (i + 1)$ (here each a, b can be any real number), $\mathfrak{X}, \mathfrak{Y}$ each has span $\mathbb{C}$. (2 pts for each $\mathfrak{X}, \mathfrak{Y}$)

Since the only real value each a, b can take so that $a \cdot 1 + b \cdot i = 0$ is 0, $\mathfrak{X}$ is linearly independent. For real numbers a, b, $a \cdot 1 + b \cdot (i + 1) = (a + b) \cdot 1 + b \cdot i$, which is 0 only when $a + b = b = 0$, i.e., $a = b = 0$. This shows $\mathfrak{Y}$ is linearly independent. (2 pts for each $\mathfrak{X}, \mathfrak{Y}$)

b. Show that the map $T : \mathbb{C} \to \mathbb{C}$ given by complex conjugation (i.e., $T(a + bi) = a - bi$ for a, b in $\mathbb{R}$) is a linear transformation. (4 pts)

We need to show T preserves addition and scalar multiplication.

For any a, b, c, d, r in $\mathbb{R}$:

$T((a + bi) + (c + di)) = T((a + c) + (b + d)i) = (a + c) - (b + d)i$ (1 pt), $T(a + bi) + T(c + di) = (a - bi) + (c - di) = (a + c) - (b + d)i$ also (1 pt), so T preserves addition.

$T(r \cdot (a + bi)) = T(ra + rbi) = ra - rbi$ (1 pt), $r \cdot T(a + bi) = r \cdot (a - bi) = ra - rbi$ also (1 pt), so T preserves scalar multiplication.

T is a linear transformation of vector spaces over $\mathbb{R}$ but NOT a linear transformation of vector spaces over $\mathbb{C}$.

? $\quad$?

c. Compute $[id_{\mathbb{C}}]_{\mathfrak{X}\mathfrak{Y}}$, $[id_{\mathbb{C}}]_{\mathfrak{Y}\mathfrak{X}}$, $[T]_{\mathfrak{X}\mathfrak{X}}$, $[T]_{\mathfrak{X}\mathfrak{Y}}$, $[T]_{\mathfrak{Y}\mathfrak{X}}$, $[T]_{\mathfrak{Y}\mathfrak{Y}}$. (12 pts)

Each of the 6 is worth 2 points.

insert into T

$$\begin{cases} id_{\mathbb{C}}(1) = 1 = 1\cdot 1 + 0\cdot i \\ id_{\mathbb{C}}(i+1) = i+1 = 1\cdot 1 + 1\cdot i \end{cases} \quad \text{(1 pt), so } [id_{\mathbb{C}}]_{\mathfrak{X}\mathfrak{Y}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ (1 pt).}$$

$$\begin{cases} id_{\mathbb{C}}(1) = 1 = 1\cdot 1 + 0\cdot (i+1) \\ id_{\mathbb{C}}(i) = i = -1\cdot 1 + 1\cdot (i+1) \end{cases} \quad \text{(1 pt), so } [id_{C}]_{\mathfrak{Y}\mathfrak{X}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \text{ (1 pt).}$$

$\mathfrak{X}=\{1,i\}$

$$\begin{cases} T(1) = 1 = 1\cdot 1 + 0\cdot i \\ T(i) = -i = 0\cdot 1 + (-1)\cdot i \end{cases} \quad \text{(1 pt), so } [T]_{\mathfrak{X}\mathfrak{X}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ (1 pt).}$$

There are 2 ways to compute the last 3 matrices.

$[T]_{\mathfrak{X}\mathfrak{Y}} = [T]_{\mathfrak{X}\mathfrak{X}}[id_{\mathbb{C}}]_{\mathfrak{X}\mathfrak{Y}}$ (1 pt) $= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ (1 pt), or

from $\begin{cases} T(1) = 1 = 1\cdot 1 + 0\cdot i \\ T(i+1) = 1 - i = 1\cdot 1 + (-1)\cdot i \end{cases}$ (1 pt) we get $[T]_{\mathfrak{X}\mathfrak{Y}} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ (1 pt).

$[T]_{\mathfrak{Y}\mathfrak{X}} = [id_{\mathbb{C}}]_{\mathfrak{Y}\mathfrak{X}}[T]_{\mathfrak{X}\mathfrak{X}}$ (1 pt)$=\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ (1 pt), or

from $\begin{cases} T(1) = 1 = 1\cdot 1 + 0\cdot (i+1) \\ T(i) = -i = 1\cdot 1 + (-1)\cdot (i+1) \end{cases}$ (1 pt) we get $[T]_{\mathfrak{Y}\mathfrak{X}} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ (1 pt).

$[T]_{\mathfrak{Y}\mathfrak{Y}} = [id_{\mathbb{C}}]_{\mathfrak{Y}\mathfrak{X}}[T]_{\mathfrak{X}\mathfrak{X}}[id_{\mathbb{C}}]_{\mathfrak{X}\mathfrak{Y}} (= [id_{\mathbb{C}}]_{\mathfrak{Y}\mathfrak{X}}[T]_{\mathfrak{X}\mathfrak{Y}} = [T]_{\mathfrak{Y}\mathfrak{X}}[id_{\mathbb{C}}]_{\mathfrak{X}\mathfrak{Y}})$ (1 pt) $= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ (1 pt), or from $\begin{cases} T(1) = 1 = 1\cdot 1 + 0\cdot i \\ T(i+1) = 1 - i = 2\cdot 1 + (-1)\cdot (i+1) \end{cases}$ (1 pt) we get $[T]_{\mathfrak{Y}\mathfrak{Y}} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ (1 pt).