Instructions

1. The statements in Italics are for introducing results and notations that may be used again in this course. You are only required to read and think about them.

2. To receive full credit you must explain how you got your answer.

3. While I encourage collaboration, you must write solutions IN YOUR OWN WORDS. DO NOT SHARE COMPLETE SOLUTIONS before they are due. YOU WILL RECEIVE NO CREDIT if you are found to have copied from whatever source or let others copy your solutions.

4. Homework must be handwritten (electronic handwriting is allowed) for authentication purposes and submitted on Canvas. Please do NOT include any personal information such as your name and netID in your file. Late homework will NOT be accepted. It is your responsibility to MAKE SURE THAT YOUR SUBMISSIONS ARE SUCCESSFUL AND YOUR FILES ARE LEGIBLE AND COMPLETE. It is also your responsibility that whoever reads your work will understand and enjoy it. Up to 4 points out of 40 may be taken off if your solutions are hard to read or poorly presented.

Homework 1 Solution

These short videos will give you the matrix background you need for some of the problems: Intro to Matrices, Operations with Matrices, How to Multpiply Matrices.

1. a.What sized matrices can be multiplied to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ on the left and right, respectively? Find ALL allowed sizes. (2 pts)

Matrices of size $n \times 2$ can be multiplied on the left, and matrices of size $2 \times n$ can be multiplied on the right. Here n can be any positive integer.

b. What do you get when you do such multiplications? Computing a few examples might help you draw the general conclusion. (2 pts)

When you multiply a matrix to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ on the left or right, the product is just your original matrix. (1 pt) Examples or explainations. (1 pt)

c. In general, the **square matrices** (*i.e. matrices with the same number of rows and columns*) with 1's on the diagonal (*when we speak of the diagonal in this class, we always mean the upper-left to lower-right diagonal*) and 0's elsewhere are called **Identity matrices**. Can you generalize your conclusion in b to all identity matrices? Give the general statement. (2 pts)

When you multiply the $m \times m$ identity matrix to a matrix of size $m \times n$ on the left, or to a matrix of size $n \times m$ on the right, the product is just the matrix being multiplied to. Here m, n can be any positive integer.

2. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$, and $C = \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix}$, where the a_{**}, b_{**}, c_{**} are real numbers. Compute $(AB)C$ and $A(BC)$ and show that they are equal. *One can generalize this and show that for general matrices A, B, and C, $(AB)C = A(BC)$ whenever the products are defined, i.e., matrix multiplication is* ***associative****. For this reason, we can use notations like A^3 and so on without causing confusion.* (4 pts)

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \end{bmatrix} \ (1pt)$$
$$(AB)C = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31})c_{11} + (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32})c_{21} \end{bmatrix} \ (1pt)$$
$$BC = \begin{bmatrix} b_{11}c_{11} + b_{12}c_{21} \\ b_{21}c_{11} + b_{22}c_{21} \\ b_{31}c_{11} + b_{32}c_{21} \end{bmatrix} \ (1pt)$$
$$A(BC) = \begin{bmatrix} a_{11}(b_{11}c_{11} + b_{12}c_{21}) + a_{12}(b_{21}c_{11} + b_{22}c_{21}) + a_{13}(b_{31}c_{11} + b_{32}c_{21}) \end{bmatrix} \ (1pt)$$

Opening the brackets one sees that the single entry in $(AB)C$ and $A(BC)$ are the same up to reordering of the terms being summed.

3. a. Show that the set of 2×3 matrices with entries in $\mathbb{R}$, together with the usual addition and scalar multiplication, is a vector space over $\mathbb{R}$. *We denote this vector space by $Mat_{2\times 3}(\mathbb{R})$.* (6 pts)

Here the two operations are given, so we just need to verify the axioms. In the following each $r, s, a_{**}, b_{**}, c_{**}$ is allowed to take any real number.

Axioms for addition (3 pts):

1)
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$$
$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} & b_{13}+a_{13} \\ b_{21}+a_{21} & b_{22}+a_{22} & b_{23}+a_{23} \end{bmatrix}$$

Since adding real numbers is commutative, the resulting matrices are equal. This shows commutativity of matrix addition.

2)
$$\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\right) + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$
$$= \begin{bmatrix} (a_{11}+b_{11})+c_{11} & (a_{12}+b_{12})+c_{12} & (a_{13}+b_{13})+c_{13} \\ (a_{21}+b_{21})+c_{21} & (a_{22}+b_{22})+c_{22} & (a_{23}+b_{23})+c_{23} \end{bmatrix}$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \left(\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}\right)$$
$$= \begin{bmatrix} a_{11}+(b_{11}+c_{11}) & a_{12}+(b_{12}+c_{12}) & a_{13}+(b_{13}+c_{13}) \\ a_{21}+(b_{21}+c_{21}) & a_{22}+(b_{22}+c_{22}) & a_{23}+(b_{23}+c_{23}) \end{bmatrix}$$

Since adding real numbers is associative, the resulting matrices are equal. This shows associativity of matrix addition.

3) Since $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the zero element/vector.

4) Since $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $-\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}$ exists for each matrix.

Axioms for scalar multiplication (3 pts):

1)$1 \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 1 \cdot a_{11} & 1 \cdot a_{12} & 1 \cdot a_{13} \\ 1 \cdot a_{21} & 1 \cdot a_{22} & 1 \cdot a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ for each matrix.

2)$(rs) \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} (rs)a_{11} & (rs)a_{12} & (rs)a_{13} \\ (rs)a_{21} & (rs)a_{22} & (rs)a_{23} \end{bmatrix}$

$r \cdot (s \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}) = r \cdot \begin{bmatrix} sa_{11} & sa_{12} & sa_{13} \\ sa_{21} & sa_{22} & sa_{23} \end{bmatrix} = \begin{bmatrix} r(sa_{11}) & r(sa_{12}) & r(sa_{13}) \\ r(sa_{21}) & r(sa_{22}) & r(sa_{23}) \end{bmatrix}$

Since multiplying real numbers is associative, the resulting matrices are equal.

3)$r \cdot (\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}) = r \cdot \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$
$= \begin{bmatrix} r(a_{11}+b_{11}) & r(a_{12}+b_{12}) & r(a_{13}+b_{13}) \\ r(a_{21}+b_{21}) & r(a_{22}+b_{22}) & r(a_{23}+b_{23}) \end{bmatrix}$

$r \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + r \cdot \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \end{bmatrix} + \begin{bmatrix} rb_{11} & rb_{12} & rb_{13} \\ rb_{21} & rb_{22} & rb_{23} \end{bmatrix}$
$= \begin{bmatrix} ra_{11}+rb_{11} & ra_{12}+rb_{12} & ra_{13}+rb_{13} \\ ra_{21}+rb_{21} & ra_{22}+rb_{22} & ra_{23}+rb_{23} \end{bmatrix}$

Since multiplication distributes over addition for real numbers, the resulting matrices are equal.

4)$(r+s) \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} (r+s)a_{11} & (r+s)a_{12} & (r+s)a_{13} \\ (r+s)a_{21} & (r+s)a_{22} & (r+s)a_{23} \end{bmatrix}$

$r \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + s \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} ra_{11}+sa_{11} & ra_{12}+sa_{12} & ra_{13}+sa_{13} \\ ra_{21}+sa_{21} & ra_{22}+sa_{22} & ra_{23}+sa_{23} \end{bmatrix}$

Since multiplication distributes over addition for real numbers, the resulting matrices are equal.

b. Describe the span of the following sets of vectors in the simplest possible terms:

i. $\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\}$. (2 pts)

The span is the set of matrices of the form $a \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, where each a, b are allowed to be any real number. (1 pt) This is the same as the set of matrices of the form $\begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \end{bmatrix}$, where b is allowed to be any real number. (1 pt)

ii. $\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\}$. (2 pts)

The span is the set of matrices of the form $a \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, where each a, b are allowed to be any real number. (1 pt) This is the same as the set of matrices of the form $\begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \end{bmatrix}$, where each a, b are allowed to be any real number. (1 pt)

iii. $\{\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & -1 & 0\\0 & 0 & 0\end{bmatrix}\}$. (2 pts)

Method 1: Since $\begin{bmatrix}1 & -1 & 0\\0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\end{bmatrix} + (-\begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\end{bmatrix})$, removing it will not change the span. (1 pt). Therefore, the span is the same as in ii. (1 pt)

Method 2: The span is the set of matrices of the form $a \cdot \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\end{bmatrix} + b \cdot \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\end{bmatrix} + c \cdot \begin{bmatrix}1 & -1 & 0\\0 & 0 & 0\end{bmatrix}$, where each a, b, c are allowed to be any real number. (1 pt) This is the same as the set of matrices of the form $\begin{bmatrix}a+c & b-c & 0\\0 & 0 & 0\end{bmatrix}$, where each a, b, c are allowed to be any real number. Since each $a+c, b-c$ are consequently allowed to be any real number, the span is the just the set of matrices of the form $\begin{bmatrix}d & e & 0\\0 & 0 & 0\end{bmatrix}$, where each d, e are allowed to be any real number. (1 pt)

c. In b, which sets of vectors are linearly dependent and which sets are linearly independent? (6 pts)

i. Removing $\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ doesn't change the span, so the set is linearly dependent.

ii. $a \cdot \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\end{bmatrix} + b \cdot \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\end{bmatrix} = \begin{bmatrix}a & b & 0\\0 & 0 & 0\end{bmatrix}$ for any real numbers a, b. For such a matrix to be the zero vector (which we found to be $\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ in part a), a, b must both be 0. Therefore, the set is linearly independent.

iii. We saw in part b that the third vector is a linear combination of the first two, therefore the set is linearly dependent.

d. Show that matrices with the additional constraint that the two entries on the 3rd column sum to 0 form a subspace of $Mat_{2\times 3}(\mathbb{R})$. (6 pts)

We need to show this subset of matrices contains the zero vector and is closed under addition and scalar multiplication.

We saw in part a that the zero vector is $\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$, whose both entries on the third column are 0 and therefore sum to 0. This shows the subset contains the zero vector. (2 pts)

Take two 2×3 matrices whose entries on the third column sum to 0. Say, the first matrix has entries $a, -a$ on the third column, the second matrix has entries $b, -b$ on the third column (here each a, b is allowed to take any real number), then the sum of the two matrices has entries $a + b, -a - b$ on the third column, which sum to 0. This shows the subset is closed under addition. (2 pts)

Take a matrix with $a, -a$ on the third column (here a is allowed to take any real number), and multiply it by a real number r. The resulting matrix has $ra, -ra$ on the third column, which sum to 0. This shows the subset is closed under scalar multiplication. (2 pts)

4. By a solution to the differential equation $2y'' + xy' - e^x y = 0$ we mean a twice-differentiable real-valued function $y(x)$ such that the equality $2y'' + xy' - e^x y = 0$ holds for any x for which $y(x)$ is defined. Show that the set of solutions to this differential equation, together with the usual addition and scalar multiplication, form a vector space over $\mathbb{R}$. *A basis for this vector space is called a fundamental set of solutions to this differential equation.* (6 pts)

We first show the set of real-valued functions, together with the usual addition and scalar multiplication, is a vector space over $\mathbb{R}$, and then show the set of solutions to the differential equation is a subspace.

To show the set of real-valued functions, together with the usual addition and scalar multiplication, is a vector space over $\mathbb{R}$ we just need to verify the axioms.

Let f, g, h be real-valued functions and a, b be real numbers. Axioms for addition: 1) $f + g = g + f$ and 2)$(f + g) + h = f + (g + h)$ are clear. 3) Let o denote the constant function with value 0, then $o + f = f$. Therefore, o is the zero vector. 4) We can take $-f$ to be the real-valued function whose value at x is the real number $-f(x)$. Then $f + (-f) = o$. Axioms for scalar multiplication: 1)$1 \cdot f = f$, 2) $(ab) \cdot f = a \cdot (b \cdot f)$, 3)$a \cdot (f + g) = a \cdot f + a \cdot g$ and 4)$(a + b) \cdot f = a \cdot f + b \cdot g$ are clear. (3 pts)

Each solution to the differential equation is a real-valued function by definition, so the set of solutions to the differential equation is a subset of the set of real-valued functions. To show this subset is a subspace, we need to show it contains o and is closed under addition and scalar multiplication.

Since $o(x), o'(x), o''(x) = 0$ for each real number x, the equality $2o'' + xo' - e^x o = 0$ holds for all real number x. This shows o is a solution to the differential equation and therefore lies in the subset. (1 pt)

Let y_1, y_2 be two solutions to the differential equation, i.e., the equality $2y_1'' + xy_1' - e^x y_1 = 0$ holds for any x for which $y_1(x)$ is defined, and the equality $2y_2'' + xy_2' - e^x y_2 = 0$ holds for any x for which $y_2(x)$ is defined. $y_1 + y_2$ is defined wherever both y_1 and y_2 are defined, and there $2(y_1 + y_2)'' + x(y_1 + y_2)' - e^x(y_1 + y_2) = (2y_1'' + xy_1' - e^x y_1) + (2y_2'' + xy_2' - e^x y_2) = 0$. This shows $y_1 + y_2$ is a solution to the differential equation. Hence the subset is closed under addition. (1 pt)

Let y be a solution to the differential equation and a be a real number. Then ay is defined wherever y is defined, and there $2(ay)'' + x(ay)' - e^x(ay) = a(2y'' + xy' - e^x y) = 0$. This shows ay is a solution to the differential equation. Hence the subset is closed under scalar multiplication. (1 pt)